Review

Sept 11-12, 2010

Theorem: Show that $\sqrt{2}$ is irrational.

Proof: Suppose not and $\sqrt{2} = \frac{p}{q}$ where the greatest common divisor of p and q is 1. Then

$$2 = \frac{p^2}{q^2} \quad \text{implies} \quad p^2 = 2q^2$$

and so p^2 is even. Since an odd number times an odd number must be again odd, it follows that p is even. Thus $p = 2k$ for some $k \in \mathbb{Z}$.

Then $(2k)^2 = 2q^2$ implies $q^2 = 2k$. It follows that q is even and so $q = 2l$ for some $l \in \mathbb{Z}$.

But $p = 2k$ and $q = 2l$ implies 2 is a common divisor of p and q which contradicts the assumption that 1 was the greatest common divisor. Therefor $\sqrt{2}$ is irrational.

True/false questions

1. If r_1 and r_2 are rational then $r_1 r_2$ is rational.

2. If a_1 and a_2 are irrational then $a_1 a_2$ is irrational.

Answers:

1. True.

2. False.

Question: Can you prove $\sqrt{3}$ is irrational?

Hint. consider $\sqrt{3} = \frac{p}{q}$ and show that p is divisible by 3.

Theorem: Between any 2 rational numbers there is an irrational number.

Proof: Let $r_1 < r_2$ be 2 rational numbers, and define

$$z = \frac{r_1 + r_2\sqrt{2}}{1+\sqrt{2}}.$$

Then z is a weighted average of r_1 and r_2 so that $r_1 < z < r_2$. Moreover,

$$z = \frac{r_1 + r_2\sqrt{2}}{1+\sqrt{2}} \cdot \frac{\sqrt{2}-1}{\sqrt{2}-1} = \frac{2r_2 - r_1 + (r_1 - r_2)\sqrt{2}}{1}$$

$$= r_3 + r_4\sqrt{2}$$

where $r_4 = r_1 - r_2 \neq 0$. Therefore z is irrational.

Theorem: Between any 2 real numbers there is an irrational number.

Proof: Read the proof in the text.

What about the proof?

It is completely different than the above proof. It is based on the fact that the interval $[0,1)$ of real numbers is uncountable. This result was proved a number of different ways by Cantor. One of the easiest proofs is by diagonalization.

Suppose that $[0,1)$ was countable. Then the numbers could be listed in order.

$$x_1 = 0.4568792 31\ldots$$
$$x_2 = 0.3889621 38\ldots$$
$$x_3 = 0.2112633 48\ldots$$
$$x_4 = 0.3449185 972\ldots$$
$$x_5 = 0.5129074 21\ldots$$
$$\vdots \quad 0.527835002\ldots$$

Take the diagonal of digits.

$$.481105\ldots$$

change

$$d = .377774$$

by the rule

digit	change
0	7
1	7
2	1
3	2
4	3
5	4
6	5
7	6
8	7
9	8

<u>Claim</u>, the number d is not in the list. It is not equal to x_1 because the first digit is different. It is not equal to x_2 because the second digit is different. It is not equal to x_n because the n^{th} digit is different. Therefore $d \neq x_n$ for all $n \in \mathbb{N}$.

However $d \in [0,1)$ and so $d = x_n$ for some n. This is a contradiction. Therefore $[0,1)$ is uncountable.

<u>But wait</u> ...

$.4999\overline{9}$ is different from $.5$ in every digit, yet these two decimals represent the same number.

<u>Review of how to sum the Geometric Series</u>

$$S = \sum_{k=p}^{q} a^k = a^p + \cancel{a^{p+1}} + \cancel{\cdots} + \cancel{a^q} \quad \text{for } p < q.$$

$$aS = \cancel{a^{p+1}} + \cancel{a^{p+2}} + \cancel{\cdots} + a^{q+1}.$$ Subtract and to get $(1-a)S = a^p - a^{q+1}$. Thus $S = \dfrac{a^p - a^{q+1}}{1-a}$.

Therefore

$$.4\overline{999} = \frac{4}{10} + \sum_{k=2}^{\infty} \frac{9}{10^k} = \frac{4}{10} + \lim_{n\to\infty} \sum_{k=2}^{n} \frac{9}{10^k}$$

Sin

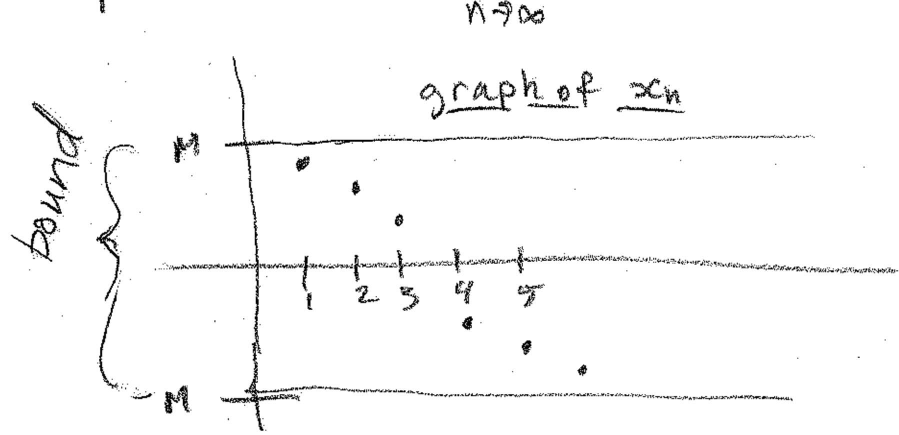
$$= \frac{4}{10} + 9 \lim_{n\to\infty} \sum_{k=2}^{n} \left(\frac{1}{10}\right)^k$$

$$= \frac{4}{10} + 9 \lim_{n\to\infty} \frac{\left(\frac{1}{10}\right)^2 - \left(\frac{1}{10}\right)^{n+1}}{1 - \frac{1}{10}}$$

$$= \frac{4}{10} + 9 \frac{\left(\frac{1}{100}\right)}{\left(\frac{9}{10}\right)} = \frac{4}{10} + \frac{1}{10} = \frac{5}{10} = .5$$

This can only happen between a decimal that has a repeating 9 pattern and a terminating decimal. This is why the rule at the bottom of page 3 doesn't map any digit to 0 or 9.

<u>Theorem</u>: If x_n is a bounded decreasing sequence then $\lim_{n\to\infty} x_n$ exists.



Try by contradiction.

Suppose $\lim_{n \to \infty} x_n$ did not exist. This

means:

there is another $\forall$ here

$$\text{NOT}\left(\exists L \in \mathbb{R} \text{ s.t. } \forall \varepsilon > 0\ \exists N > 0 \text{ s.t. } \underbrace{n \geq N \text{ implies}}_{\forall n \geq N \text{ then}} |x_n - L| < \varepsilon\right)$$

To negate a proposition with universal and existential quantifiers in is

$\exists$ changes to $\forall$

$\forall$ changes to $\exists$

Thus not converging means

$$\forall L \in \mathbb{R}\ \exists\, \varepsilon > 0 \text{ s.t. } \forall N > 0\ \exists n \geq N \text{ s.t. } |x_n - L| \geq \varepsilon.$$

It sure would be nice to know how to choose L to get a contradiction.

Look at the picture and see the x_n appear to be converging to the lower bound. Thus take

$$L = \inf\{x_n : n \in \mathbb{N}\}$$

the greatest lower bound.

Then

$\exists \varepsilon > 0$ st $\forall N > 0$ $\exists n_0 \geq N$ st $|x_{n_0} - L| \geq \varepsilon$.

Need to use that x_n is decreasing since otherwise the result is obviously not true. How?

Decreasing means $x_n \geq x_{n+1}$. Thus $n_0 \geq N$ implies that $x_N \geq x_{n_0}$. Thus

$$|x_{n_0} - L| \geq \varepsilon \quad \text{for some } n_0 \geq N.$$

and L a lower bound means $|x_n - L| = x_n - L$, so

$$x_{n_0} - L \geq \varepsilon \quad \text{for some } n_0 \geq N.$$

or $\quad x_{n_0} \geq \varepsilon + L$

Thus $\forall N > 0$ $\exists n_0 \geq N$ such that

$$x_N \geq x_{n_0} \geq \varepsilon + L, \text{ which}$$

implies $x_N \geq \varepsilon + L$ for all $N > 1$. Therefore

Therefor $\varepsilon + L$ is a lower bound for $\{x_n : n \in \mathbb{N}\}$. But $\varepsilon + L > L$ contradicts that L was the greatest lower bound.

It follows $\lim_{n \to \infty} x_n = L = \inf\{x_n : n \in \mathbb{N}\}$.

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