Math 713 Summary Sept 13, '10

Equivalent definitions:

Cluster Point of a sequence:

Let $x_n \in \mathbb{R}$, then x is a cluster point of x_n if there is a subsequence x_{n_k} such that $x_{n_k} \to x$.

equivalently:

$x \in \mathbb{R}$ is a cluster point if $\forall \varepsilon > 0$ and $\forall N > 0$ $\exists n \geq N$ s.t. $|x_n - x| < \varepsilon$.

Note that $x = \infty$ is a cluster point if $\forall M > 0$ and $\forall N > 0$ $\exists n \geq N$ such that $x_n > M$,

and $x = -\infty$ is a cluster point if $\forall M > 0$ and $\forall N > 0$ $\exists n \geq N$ such that $x_n > M$.

Limit point of a set:

Let $A \subseteq \mathbb{R}$ be a set. Then x is a cluster point of A if there is a sequence $x_n \in A$ such that $x_n \to x$.

equivalently:

$x \in \mathbb{R}$ is a limit point if $\forall \varepsilon > 0$ $\exists y \in A$ s.t. $|x-y| < \varepsilon$.

Again the case $x = \infty$ and $x = -\infty$ have to be treated separately.

In particular $x = \infty$ is a limit point if $\forall M > 0$ $\exists y \in A$. s.t. $y > M$,

and $x = -\infty$ is a limit point if $\forall M > 0$ $\exists y \in A$ s.t. $y < -M$.

Accumulation point of a set

Let $A \subseteq \mathbb{R}$. Then x is an accumulation point of A if there is a sequence $x_n \in A$ of distinct points $x_n \in A$ such that $x_n \to x$.

equivalently

$x \in \mathbb{R}$ is an accumulation point if $\forall \varepsilon > 0$ $\exists y \in A$ s.t. $0 < |x - y| < \varepsilon$,

$x = \infty$ is an ——— " ——— $\forall M > 0$ $\exists y \in A$ s.t. $y > M$, and

$x = -\infty$ is an ——— " ——— $\forall M > 0$ $\exists y \in A$ s.t. $y < -M$.

Each term is defined in 2 ways. How hard is it to show these definitions are equivalent?

Let $E \subseteq \mathbb{R}$. Define

$$\overline{E} = \{x \in \mathbb{R} : x \text{ is a limit point}\}.$$

The set $\overline{E}$ is called the closure of E.

<u>Equivalent definitions</u> for a set $A \subseteq \mathbb{R}$ to be <u>closed</u>.

$A \subseteq \mathbb{R}$ is closed if $\overline{A} = A$

<u>equivalently</u>

$A \subseteq \mathbb{R}$ is closed if $\mathbb{R} \setminus A$ is open.

<u>Recall</u> <u>definition of open</u>

$\vartheta \subseteq \mathbb{R}$ is open if $\forall x \in \vartheta\ \exists \varepsilon > 0$ s.t. $|x-y| < \varepsilon$ implies $y \in \vartheta$.

Equivalent definitions of limsup x_n

Let $x_n \in \mathbb{R}$ be a sequence. Then

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_k : k>n\}$$

equivalently

$$\limsup_{n\to\infty} x_n = \sup\{x \in \mathbb{R}^* : x \text{ is a cluster point of } x_n\}$$

Example:

Let $x_n = (-1)^n$. Then

$$y_n = \sup\{x_k : k>n\} = \sup\{1,-1\} = 1$$

so

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 1.$$

alternatively we find the set of all cluster points of x_n and take the supremum of this set as follows:

Recall that x_{n_k} is a subsequence if $n_k \in \mathbb{N}$ and $n_k < n_{k+1}$ for all $k \in \mathbb{N}$.

Thus $n_k = 2k$ defines a subsequence and

$$x_{n_k} = (-1)^{n_k} = (-1)^{2k} = 1 \to 1.$$

So 1 is a cluster point of x_n.

Similarly, $n_k = 2k+1$ is another subsequence and

$$x_{n_k} = (-1)^{n_k} = (-1)^{2k+1} = -1 \to -1$$

So -1 is also a cluster point.

<u>Claim</u> there are no other cluster points of this sequence.

For contradiction suppose there were a subsequence $x_{n_k} \to L$ where $L \notin \{-1, 1\}$.

Define

$$\varepsilon = \frac{1}{2}\min\{|L-1|, |L+1|\} > 0$$

and choose $K > 0$ so that $k \geq K$ implies $|x_{n_k} - L| < \varepsilon$.

Case $x_{n_k} = 1$. Then

$$|x_{n_k} - L| = |1 - L| \geq 2\varepsilon$$

contradicts $|x_{n_k} - L| < \varepsilon$. Therefore this case can't happen.

Case $x_{n_k} = -1$. Then

$$|x_{n_k} - L| = |1 + L| \geq 2\varepsilon$$

is also a contradiction.

Since $x_n = (-1)^n \in \{-1, 1\}$ these are the only cases to consider. Therefore the set of cluster points is $\{-1, 1\}$.

It follows that

$$\limsup_{n\to\infty} x_n = \sup\left\{x\in\mathbb{R}^* : \begin{array}{l} x \text{ is a cluster} \\ \text{point of } x_n \end{array}\right\}$$

$$= \sup\{-1,1\} = 1.$$

Equivalent definitions of $\liminf x_n$

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf\{x_k : k \geq n\}.$$

equivalently

$$\liminf_{n\to\infty} x_n = \inf\left\{x\in\mathbb{R}^* : \begin{array}{l} x \text{ is a cluster} \\ \text{point of } x_n \end{array}\right\}.$$

Please check all these definitions and try to show they are equivalent.