Review Summary

Sept 18, '10

Discussion of homework problem 6

Let $f:[0,1]\to\mathbb{R}$ be continuous and $c\in(0,1)$ such that $f(c)>0$. Then there exists $h>0$ such that $|x-c|<h$ implies $f(x)>0$.

Ideas? If stuck try an example.

$f(x)=x^3$ and $f(\frac{1}{2})=\frac{1}{8}>0$.

Now find the h so that $|x-\frac{1}{2}|<h$ implies $x^3>0$

obviously $h=\frac{1}{2}$ works, because

$$|x-\tfrac{1}{2}|<\tfrac{1}{2}$$

implies

$$-\tfrac{1}{2}<x-\tfrac{1}{2}<\tfrac{1}{2}$$

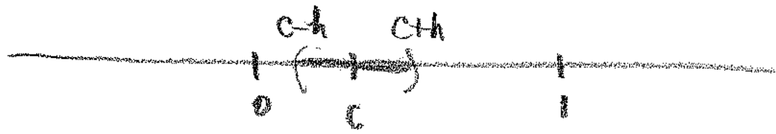
So $0<x<1$. Since $x>0$ then $x^3>0$.

Try to generalize example.

Suppose we take general $c \in (0,1)$.

Then $f(c) = c^3 > 0$

Now find h.



Then

$$|x-c| < h$$

implies

$$-h < x-c < h$$

So $$c-h < x < h+c$$

Thus for $x \in (0,1)$ we need

$$c-h > 0 \quad \text{and} \quad c+h < 1$$

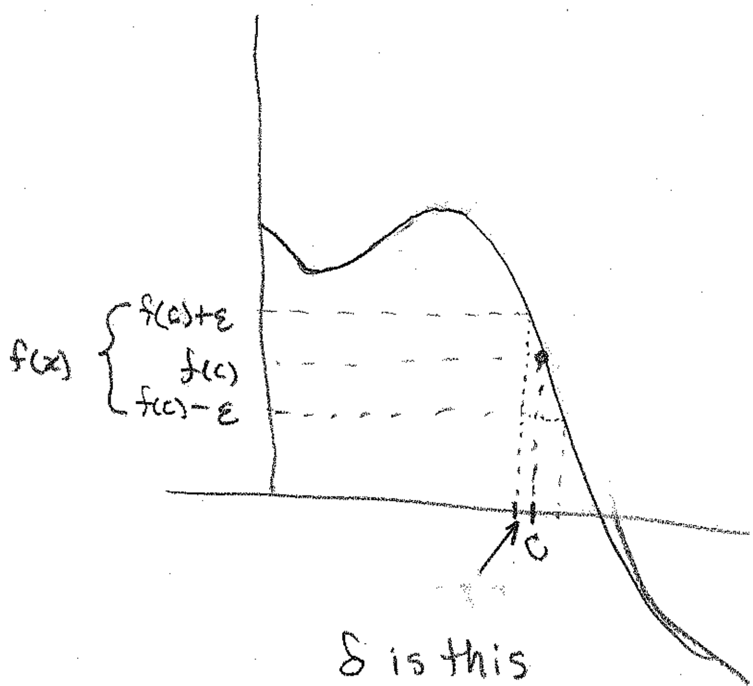
or $$h < c \quad \text{and} \quad h < 1-c$$

Taking $h = \min(c, 1-c)$ gives a value such that $|x-c| < h$ implies $f(x) > 0$.

What if f is not exactly known?

Then we need to use the continuity of f instead of the fact that x^3 is positive exactly when x is positive.

If f is continuous on $[0,1]$ then for every $\varepsilon > 0$ there is $\delta > 0$ such that $|x-c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$.



δ is this distance.

Now

$$|f(x) - f(c)| < \varepsilon$$

implies

$$-\varepsilon < f(x) - f(c) < \varepsilon$$

$$f(c) - \varepsilon < f(x) < f(c) + \varepsilon$$

Now what?

If stuck go back to $f(x) = x^3$. Thus

$$c^3 - \varepsilon < x^3 < c^3 + \varepsilon$$

and taking $\varepsilon = \frac{1}{2}c^3$ gives that

$$x^3 > \frac{1}{2}c^3 > 0 \text{ for } |x - c| < \delta.$$

How to choose ε in general?

Try $\varepsilon = \frac{1}{2}f(c) > 0$ since $f(c) > 0$. Thus

$$f(c) - \frac{1}{2}f(c) < f(x) < f(c) + \frac{1}{2}f(c)$$

implies $f(x) > \frac{1}{2}f(c) > 0$ provided x is in the domain of f and $|x - c| < \delta$.

To finish the problem find a way of choosing h so

$$|x-c| < h$$

implies that both $x \in [0,1]$

and $|x-c| < \delta$.

The second part of problem 6 can be obtained using proof by contradiction and the result from the first part.

<u>Review of δ-ε arguments</u>

Show that $\lim_{x \to c} x^3 = c^3$.

Let $\varepsilon > 0$ and choose $\delta =$ "something" > 0 so that $0 < |x-c| < \delta$ implies $|x^3 - c^3| < \varepsilon$.

The idea is to estimate $|x^3 - c^3|$ in terms of δ and the choose the "something" for δ so that one gets an ε for the bound.

In particular,

Suppose $0<|x-c|<\delta$. Then

$$|x^3-c^3| = |x-c|\,|x^2+2xc+c^2|$$
$$\leq \delta\,|x^2+2xc+c^2|$$
$$\leq \delta\left(|x|^2+2|x||c|+|c|^2\right)$$

Now $|x-c|<\delta$ implies

$$-\delta < x-c < \delta$$
$$c-\delta < x < c+\delta$$
$$-|c|-\delta < x < |c|+\delta$$

so $$|x|<|c|+\delta$$

Therefore

$$|x^3-c^3| < \delta\left((|c|+\delta)^2+2(|c|+\delta)|c|+|c|^2\right)$$
$$< 3\delta\,(|c|+\delta)^2$$

Now that we have a bound on $|x^3 - c^3|$ that depends on δ but not on x we can find a δ so

$$3\delta\left(|c|+\delta\right)^2 \leq \varepsilon ,$$

This is a polynomial inequality of the form $p(\delta) \leq 0$ where

$$p(\delta) = 3\delta(|c|+\delta)^2 - \varepsilon ,$$

In algebra class we solve by finding the roots of $p(\delta)$ and making a table

$$- \qquad + \qquad - \qquad +$$

and choosing the intervals where the polynomial is negative

In this case we do not need the full solution of the inequality $p(\delta) \leq 0$ but only one value of $\delta > 0$ where $p(\delta) \leq 0$.

Here is one method to obtain such a value of δ:

$$3\delta(|c|+\delta)^2 \leq \varepsilon$$

There are two δ's in the problem, Treat each separately and combine the conditions by taking a minimum.

Estimate the second δ by $\delta \leq 1$.

$$3\delta(|c|+\delta)^2 \leq 3\delta(|c|+1)^2$$

Estimate the first δ by $\delta \leq \dfrac{\varepsilon}{3(|c|+1)^2}$.

$$3\delta(|c|+1)^2 \leq \varepsilon$$

So choose $\delta = \min\left(1, \dfrac{\varepsilon}{3(|c|+1)^2}\right)$

Write out a clean copy of the argument

Show that $\lim_{x \to c} x^3 = c^3$.

Let $\varepsilon > 0$ and choose $\delta = \min\left(1, \frac{\varepsilon}{3(|c|+1)^2}\right)$.

Then $0 < |x-c| < \delta$ implies

$-1 < x - c < 1$ so $c-1 < x < c+1$

so $-|c|-1 < x < |c|+1$

so $|x| < |c|+1$.

It follows that

$$\begin{aligned}
|x^3 - c^3| &= |x-c|\,|x^2+xc+c^2| \\
&< \delta\,|x^2+xc+c^2| \\
&\leq \delta\left(|x|^2 + |x||c| + |c|^2\right) \\
&\leq \delta\left((|c|+1)^2 + (|c|+1)|c| + |c|^2\right) \\
&< 3\delta\,(|c|+1)^2 \leq \varepsilon.
\end{aligned}$$

Hint for problem 1.

If X is finite then any collection D of subsets of X must also be finite.

Hint 2.

Let $n_k \in \mathbb{N}$ be a monotone non-increasing sequence such that $n_k \geq n_{k+1}$. Then n_k is eventually constant.

Hint 3.

True or false any monotone sequence of sets in D is eventually constant.