Math 713 Summary Sept 22, '10

What is the definition of a function?

A function $f: A \to B$ is a subset of $A \times B$ such that $(a_1, b_1) \in A \times B$ and $(a_2, b_2) \in A \times B$ with $a_1 = a_2$ implies $b_1 = b_2$.

Note we are defining functions to be graphs that pass the vertical line test.

Functions defined by graphs can be more general than polynomials or other functions give by algebraic formulae.

Even if all you start with is polynomials it is possible to get more general functions by taking the limits of sequences of polynomials.

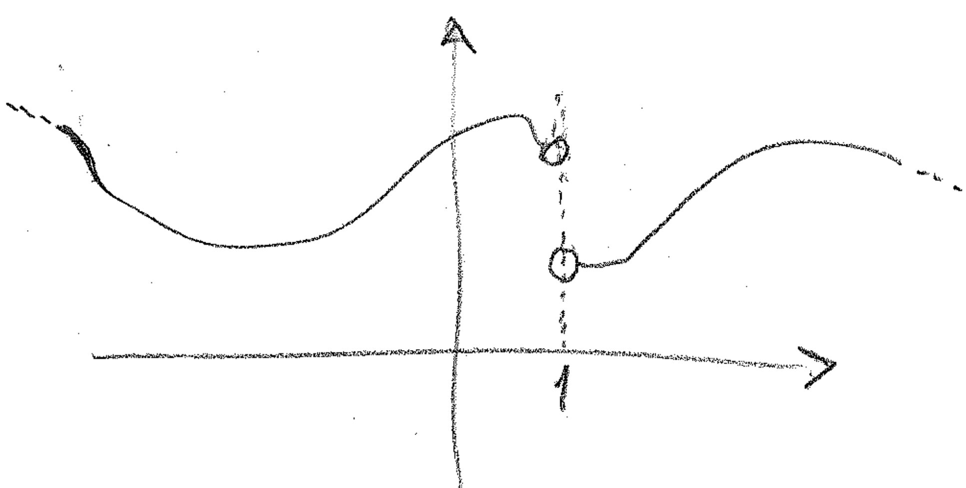
Definition of Continuity

$f : D \to \mathbb{R}$ is continuous

if and only if

$\forall x_0 \in D$ and $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $x \in D$ and $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$.

For example



Is a continuous function. To see this we need to check the ε-δ definition for every x_0 in the domain.

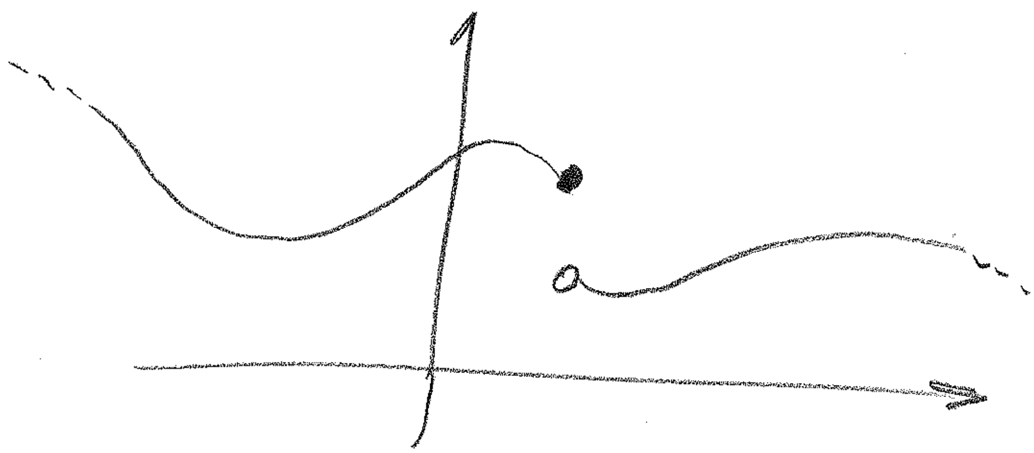
$$\text{Domain} f = D = \{x : (x,y) \in f\}.$$

It is the set of all x values of the pairs (x,y) in the graph of f. In this case

$$D = (-\infty, 1) \cup (1, \infty).$$

Since $1 \notin D$ we don't have to check ε-δ for $x_0 = 1$. Thus the funtion is continuous.

Another example



Is not a continuous function because 1 is now included in the domain and there is a jump discontinuity there.

In Calculus you might have been told.

> a function is continuous if you can draw its graph without lifting your pencil.

What needs to be assumed for this definition to be true?

> If the domain of f is $\mathbb{R}$ it is true.

Not lifting the pencil has something to do with connectedness which follows from the completeness axiom.

The open set way of defining continuity:

$f: D \to \mathbb{R}$ is continuous

if and only if

$\forall$ open set $O \subseteq \mathbb{R}$
then $f^{-1}(O)$ is open relative to D.

Is this definition easier?

At least the ε-δ went away.

But wait...

A set $G \subseteq D \subseteq \mathbb{R}$ is said to be open relative to D if for every $x_0 \in G$ there is $r > 0$ such that $(x_0 - r, x_0 + r) \cap D \subseteq G$.

So, the ε and δ are in the definition of open set masquerading as r.

Take a 5 minute break, and then we will prove that the ε-δ definition of continuity is the same as the definition using open sets.

Equivalence of the ε-δ definition of continuity to to open set definition of continuity

"$\Leftarrow$" Suppose $f^{-1}(O)$ is open in D for every open $O \subseteq \mathbb{R}$.
Let $\varepsilon > 0$ and $x_0 \in D$. Take $O = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$.
By hypothesis $f^{-1}(O)$ is open relative to D.
Since $f(x_0) \in O$ then $x_0 \in f^{-1}(O)$.
Since $f^{-1}(O)$ is open relative to D there is $r > 0$ such that $(x_0 - r, x_0 + r) \cap D \subseteq f^{-1}(O)$.
Take $\delta = r$. Then if $x \in D$ and $|x - x_0| < \delta = r$ we have $x \in f^{-1}(O)$ and thus $f(x) \in O$.
Now $f(x) \in O = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ implies $|f(x) - f(x_0)| < \varepsilon$.

"$\Rightarrow$" Suppose $\forall \varepsilon > 0, x_0 \in D \; \exists \delta > 0$ st $x \in D$ and $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$.
Let $O \subseteq \mathbb{R}$ be open. Claim $f^{-1}(O)$ is open relative D.
Let $x_0 \in f^{-1}(O)$. Then $f(x_0) \in O$. Since O is open there is $r > 0$ such that $(f(x_0) - r, f(x_0) + r) \subseteq O$. Let $\varepsilon = r$.
Then there is $\delta > 0$ such that $x \in D$ and $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$. Thus

$$f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \text{ for all } x \in D \text{ with } |x - x_0| < \delta$$

and so

$$f(x) \in O \text{ for all } x \in D \text{ with } |x - x_0| < \delta.$$

or $\quad x \in f^{-1}(O)$ for all $x \in D$ with $|x - x_0| < \delta$

or $\quad (x_0 - \delta, x_0 + \delta) \cap D \subseteq f^{-1}(O)$, which shows $f^{-1}(O)$ is open.

Definition of a Characteristic function

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Definition of a Step function $h : [a,b] \to \mathbb{R}$.

$$h(x) = \sum_{k=1}^{n} a_k \chi_{I_k}(x)$$

where $n \in \mathbb{N}$ and I_k is a collection of disjoint intervals such that

$$[a,b] = \bigcup_{k=1}^{n} I_k$$

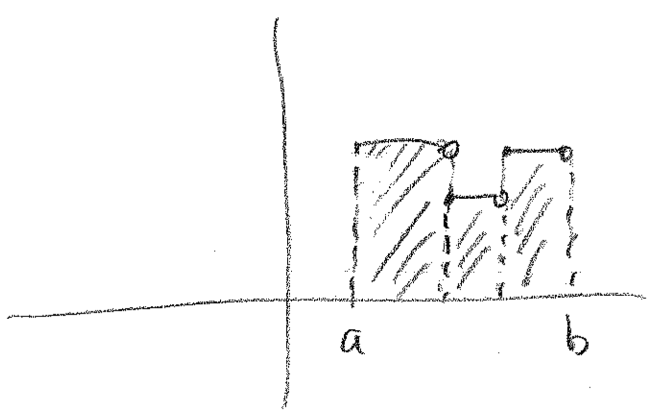
Definition of the length of an Interval

$$\ell([a,b]) = \ell((a,b)) = \ell((a,b]) = \ell([a,b)) = b-a.$$

Definition of the integral of a step function

$$\int_a^b h = \sum_{k=1}^{n} a_k \, \ell(I_k)$$

Thus



$\int_a^b h =$ sum of the areas of the rectangles.

Definition of Riemann Integral

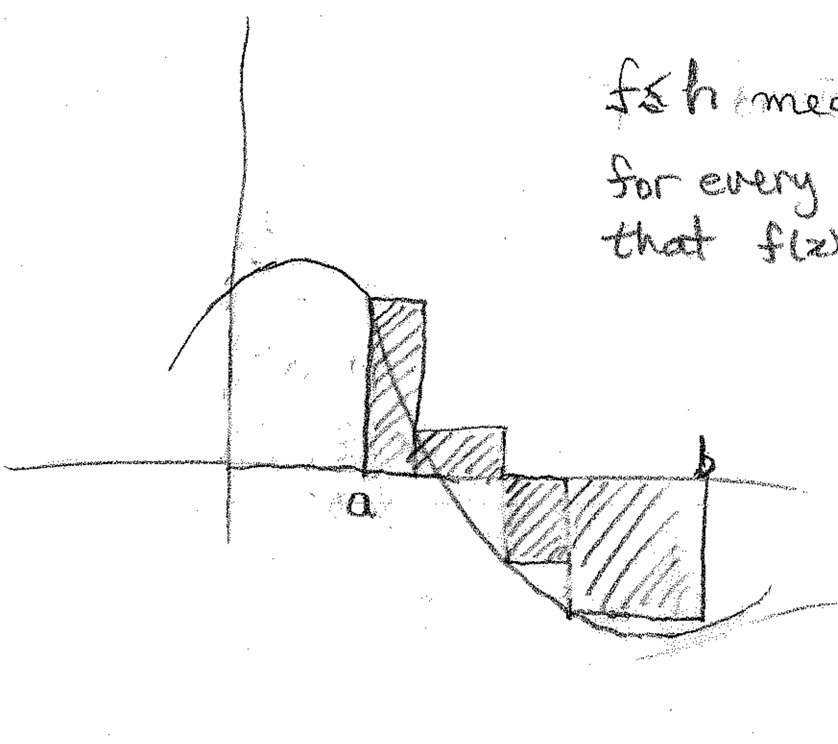
The upper Riemann integral is

$$\overline{\int_a^b} f = \inf \left\{ \int_a^b h : \begin{array}{l} h \text{ is a step function} \\ \text{and } f \le h \end{array} \right\}$$

The lower Riemann integral is

$$\underline{\int_a^b} f = \sup \left\{ \int_a^b h : \begin{array}{l} h \text{ is a step function} \\ \text{and } h \le f \end{array} \right\}$$

Thus $\overline{\int_a^b} f$ is the "smallest" of things that are too big and $\underline{\int_a^b} f$ is the "largest" of things that are too small.



$f \le h$ means for every $x \in [a,b]$ that $f(x) \le h(x)$

Let $f:[a,b]\to\mathbb{R}$ be bounded. Then f is Riemann integrable if

$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

and in which case we define the Riemann integral

$$\int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b} f .$$

<u>Definition of Lebesgue Outer Measure</u>

Let $A \subseteq \mathbb{R}$ then

$$\lambda^*(A) = \inf\left\{ \sum_{k=1}^{\infty} \ell(I_k) : A \subseteq \bigcup_{k=1}^{\infty} I_k \text{ and } I_k \text{ are open intervals} \right\}.$$

<u>Main theorem in Riemann integration</u>

Let $f:[a,b]\to\mathbb{R}$ be bounded and A the set of points of discontinuities of f. Then f is Riemann integrable if and only if $\lambda^*(A)=0$.

<u>Remark</u>

$$A = \left\{ x_0 \in [a,b] : \lim_{x\to x_0} f(x) \neq f(x_0) \right\}$$

Definition

$$\limsup_{x \to x_0} f(x) = \lim_{\varepsilon \to 0} \sup \{ f(x) : 0 < |x - x_0| < \varepsilon \}$$

$$\liminf_{x \to x_0} f(x) = \lim_{\varepsilon \to 0} \inf \{ f(x) : 0 < |x - x_0| < \varepsilon \}.$$

Is there another name for these limits?

Using the same proof as for sequences it is possible to show that the limsup and liminf of a bounded function always exist.

If $\limsup_{x \to x_0} f(x) = \liminf_{x \to x_0} f(x)$

then the limit $\lim_{x \to x_0} f(x)$ exists

and is equal to the common value of limsup and liminf.

If $\lim_{x \to x_0} f(x) = L$ exists but $L \neq f(x_0)$ then the discontinuity at x_0 is said to be removable.

Let $f : [a,b] \to \mathbb{R}$ and B the set of removable discontinuities. Is B countable?