Addendum to page 16 from Sept 24

Let $E \subseteq \mathbb{R}$ and

$$E' = \{x \in \mathbb{R} : \forall \varepsilon > 0 \; \exists y \in E \text{ s.t. } 0 < |y - x| < \varepsilon\}.$$

Claim $E' = \overline{E'}$. By definition

$$\overline{E'} = \{x \in \mathbb{R} : \forall \varepsilon > 0 \; \exists z \in E' \text{ s.t. } |z - x| < \varepsilon\}.$$

Let $x \in \overline{E'}$. Claim $x \in E'$. Let $\varepsilon > 0$.

Since $x \in \overline{E'}$ then for $\varepsilon_2 = \varepsilon/2 > 0$
there is $z \in E'$ s.t. $|z - x| < \varepsilon_2$

Case $z \neq x$.

Since $z \in E'$ then for $\varepsilon_3 = \frac{1}{2}|z - x| > 0$
there is $y \in E$ s.t. $0 < |y - z| < \varepsilon_3$.

Now $|x - y| \le |x - z| + |z - y| < \varepsilon_2 + \varepsilon_3 < \frac{\varepsilon}{2} + \frac{1}{2}|z - x|$
$< \frac{\varepsilon}{2} + \frac{1}{2}\frac{\varepsilon}{2} = \frac{3}{4}\varepsilon < \varepsilon$.

and $|x - y| \ge |x - z| - |z - y| = 2\varepsilon_3 - |z - y|$
$> 2\varepsilon_3 - \varepsilon_3 = \varepsilon_3 > 0$

implies $x \in E'$.

Case $z = x$

Since $z \in E'$ then for $\varepsilon_3 = \varepsilon > 0$
there is $y \in E$ s.t. $0 < |y - z| < \varepsilon_3$

But since $x = z$ this means $0 < |y - x| < \varepsilon$.
Therefore $x \in E'$.

Thus E' is closed.