Review Summary Oct 1, '10

Given $A, B \subseteq \mathbb{R}$ define $A \cdot B = \{ab : a \in A \text{ and } b \in B\}$.
Prove or disprove the claim that $\overline{A} \cdot \overline{B} = \overline{A \cdot B}$.

How many people tried to prove this?

Let $x \in \overline{A} \cdot \overline{B}$ then $x = a \cdot b$ where $a \in \overline{A}$ and $b \in \overline{B}$.
Thus there exist $a_n \in A$ and $b_n \in B$ such that

$$a_n \to a \quad \text{and} \quad b_n \to b.$$

Let $x_n = a_n b_n$. Then $x_n \in A \cdot B$. Moreover

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} a_n b_n = \left(\lim_{n \to \infty} a_n\right)\left(\lim_{n \to \infty} b_n\right)$$

$$= ab = x$$

implies that $x \in \overline{A \cdot B}$. Thus $\overline{A} \cdot \overline{B} \subseteq \overline{A \cdot B}$.

How to prove the reverse inclusion?
(Hint: it is not true)

Let $x \in \overline{A \cdot B}$. Then there exists $x_n \in A \cdot B$ such that $x_n \to x$. Since $x_n \in A \cdot B$ there is $a_n \in A$ and $b_n \in B$ such that $x_n = a_n b_n$

If $\lim_{n\to\infty} a_n b_n$ exists is it necessarily the case that $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist?

Example: $a_n = \frac{1}{n}$ $\quad b_n = n$

Then $\lim b_n$ doesn't exist but $\lim a_n b_n = 1$.

The problem appears to be that b_n is unbounded and tends to ∞.

Can we add conditions to avoid this problem? For example,

If a_n and b_n are bounded and $\lim_{n\to\infty} a_n b_n$ exists is it necessarily true $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist?

Example: $a_n = \dfrac{1}{14 + \sin n}$ $\quad b_n = 14 + \sin n$.

Then neither $\lim a_n$ nor $\lim b_n$ exist but $\lim a_n b_n = 1$

Since a_n and b_n are bounded then there is a subsequence such that a_{n_k} converges. Clearly b_{n_k} also converges on this subsequence. Let

$$a = \lim_{k\to\infty} a_{n_k} \quad \text{and} \quad b = \lim_{k\to\infty} b_{n_k}.$$

Is there some subsequence so that the product $ab = 1$.

Either this is obvious, dependent on properties of $\sin n$, or generally not true. So which?

Obvious since

$$ab = \lim_{k\to\infty} a_{n_k} \lim_{k\to\infty} b_{n_k} = \lim_{k\to\infty} a_{n_k} b_{n_k}$$

$$= \lim_{k\to\infty} 1 = 1.$$

Everything is now been discussed to prove the result.

Given $A, B \subseteq \mathbb{R}$ such that A and B are bounded then $\overline{A} \cdot \overline{B} = \overline{A \cdot B}$.

Take a moment to write the proof. Note that bounded is essential because of the counter example that is presented in the practive quiz 1. solutions. Done?

We have already shown $\overline{A} \cdot \overline{B} \subseteq \overline{A \cdot B}$ without the additional hypothesis of A and B being bounded.

Let $x \in \overline{A \cdot B}$. Then there exists $x_n \in A \cdot B$ such that $x_n \to x$. Since $x_n \in A \cdot B$ there is $a_n \in A$ and $b_n \in B$ such that $x_n = a_n b_n$.

Since a_n is bounded there is a subsequence a_{n_k} such that $a_{n_k} \to a$ for some $a \in \overline{A}$.

Since b_{n_k} is bounded there is a sub-subsequence $b_{n_{k_j}}$ such that $b_{n_{k_j}} \to b$ for some $b \in \overline{B}$.

Now

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} a_n b_n = \lim_{j\to\infty} a_{n_{k_j}} b_{n_{k_j}}$$

$$= \left(\lim_{j\to\infty} a_{n_{k_j}}\right)\left(\lim_{j\to\infty} b_{n_{k_j}}\right) = ab \in \overline{A}\cdot\overline{B}$$

shows that $x \in \overline{A}\cdot\overline{B}$. Thus $\overline{A\cdot B} \subseteq \overline{A}\cdot\overline{B}$.

Did, your proof have the subsubsequence? Is it really necessary or does b_{n_k} already converge because a_{n_k} did?

Thus, if $\lim a_k b_k$ and $\lim a_k$ exist then is it necessarily true that $\lim b_k$ exists?

If you were in calculus how would you solve this?

Mathematics used to be about computing things. Now it seems we want graduate students to write down nice proofs from the beginning. Many proofs are based on a computation.

In Calculus we would write

$$\lim a_k b_k = x$$

$$\lim a_k = a$$

then

$$\lim b_k = \lim \frac{a_k b_k}{a_k} = \frac{\lim a_k b_k}{\lim a_k} = \frac{x}{a}$$

Therefore, $\lim b_k$ exists and is equal $\frac{x}{a}$.

The computation also tells us that the limit doesn't exist if $a=0$. Thus the sub-subsequence is necessary to take care of the case where $a=0$.

Alternatively we could write

If $A, B \subseteq \mathbb{R}$ where A is bounded and $0 \notin \bar{A}$ then $\bar{A} \cdot \bar{B} = \overline{A \cdot B}$.

How do you show the product of two continuous functions is continuous?

In Calculus we used the ε-δ definition of continuity. Is it easier to use the open set definition?

$f: D \to \mathbb{R}$ is continuous if O open in $\mathbb{R}$ implies $f^{-1}(O)$ is open in D.

We need to show $(fg)^{-1}(O)$ is open in D.

$$(fg)^{-1}(O) = \{x \in D : f(x)g(x) \in O\}$$

How to show this is open?

It seems difficult. Can you remember the δ-ε proof from Calculus?

Let $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$,

Suppose $\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$ for $a \in D$. Then $\lim_{x \to a} (fg)(x) = (fg)(a)$.

Proof: Let $\varepsilon > 0$.

Choose $\varepsilon_2 = \boxed{} > 0$. Then since $\lim_{x \to a} f(x) = f(a)$ there is $\delta_2 > 0$ such that $x \in D$ and $|x-a| < \delta_2$ implies $|f(x) - f(a)| < \varepsilon_2$

Choose $\varepsilon_3 = \boxed{} > 0$. Then since $\lim_{x \to a} g(x) = g(a)$ there is $\delta_3 > 0$ such that $x \in D$ and $|x-a| < \delta_3$ implies $|g(x) - g(a)| < \varepsilon_3$.

Let $\delta = \min(\delta_2, \delta_3)$. Then $x \in D$ and $|x-a| < \delta$ implies

$$|f(x)g(x) - f(a)g(a)| \leq |f(x)g(x) - f(a)g(x)| + |f(a)g(x) - f(a)g(a)|$$

$$< |g(x)|\,\varepsilon_2 + |f(a)|\,\varepsilon_3$$

$$\leq (|g(x) - g(a)| + |g(a)|)\,\varepsilon_2 + |f(a)|\,\varepsilon_3$$

$$\leq (\varepsilon_3 + |g(a)|)\,\varepsilon_2 + |f(a)|\,\varepsilon_3.$$

Now it is a matter of choosing ε_2 and ε_3 in such a way that

$$(\varepsilon_3 + |g(a)|)\varepsilon_2 + |f(a)|\varepsilon_3 \leq \varepsilon.$$

Take a few minutes to find a nice choice of ε_2 and ε_3 that do this.

Here is one way...

There are ε_3 appears in two places estimate each one seperately and then combine the restrictions on the size of ε_3 by taking a minimum.

In particular let $\varepsilon_3 = \min(1, \boxed{\quad})$ so that

$$(\varepsilon_3 + |g(a)|)\varepsilon_2 + |f(a)|\varepsilon_3$$

$$\leq (1 + |g(a)|)\varepsilon_2 + |f(a)|\varepsilon_3.$$

Now it is easy to see what ε_2 needs to be so the first term is less than $\varepsilon/2$.

Take $\varepsilon_2 = \dfrac{\varepsilon}{2(1+|g(a)|)}$

For the second term we need

$$\varepsilon_3 \leq \frac{\varepsilon}{2|f(a)|}$$

So take $\varepsilon_3 = \min\left(1, \dfrac{\varepsilon}{2|f(a)|}\right)$. It follows that

$$|f(x)g(x) - f(a)g(a)| \leq (\varepsilon_3 + |g(a)|)\varepsilon_3 + |f(a)|\varepsilon_3$$

$$\leq (1+|g(a)|)\,\varepsilon_3 + |f(a)|\,\varepsilon_3$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

If you were grading this solution is there anything you might be concerned about and take off points for?

What about $f(a)$ when it is zero?

Then $\dfrac{\varepsilon}{2|f(a)|}$ doesn't make sense...

One way to fix this is by adding 1 to the denominator to get

$$\varepsilon_3 = \min\left(1, \frac{\varepsilon}{2(1+|f(a)|)}\right).$$

In this case

$$(1+|g(a)|)\varepsilon_3 + |f(a)|\varepsilon_3$$

$$\le \frac{\varepsilon}{2} + \frac{|f(a)|}{1+|f(a)|}\frac{\varepsilon}{2} \le \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon,$$

and the proof is finished.

Think a little about what it really means if $f(a)=0$ in the proof.

Is it good to combine the case $f(a)=0$ with the case $f(a)\neq 0$ at the expense of getting a sharper estimate on how big ε_3 can be? Under what circumstance might you want to work the problem case by case?