Review Summary

Oct 2, '10

Let $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$. Let $a \in D$. If

$$\lim_{x\to a} f(x) = f(a) \text{ and } \lim_{x\to a} g(x) = g(a)$$

then

$$\lim_{x\to a} (f(x)+g(x)) = f(a)+g(a)$$

and

$$\lim_{x\to a} (f(x)g(x)) = f(a)g(a).$$

Either look up the answer in your Calculus book or figure out the proof. The way to write the proof is different if you are figuring it out.

Let $\varepsilon > 0$ and $a \in D$.

Choose $\varepsilon_1 = \boxed{}$. Then there is $\delta_1 > 0$ such that $x \in D$ and $0 < |x-a| < \delta_1$ implies $|f(x)-f(a)| < \varepsilon_1$.

Choose $\varepsilon_2 = \boxed{}$. Then there is $\delta_2 > 0$ such that $x \in D$ and $0 < |x-a| < \delta_2$ implies $|g(x)-g(a)| < \varepsilon_2$.

Choose $\delta = \min(\delta_1, \delta_2)$. Then $x \in D$ and $0 < |x-a| < \delta$ implies

$$\begin{aligned} |f(x)+g(x)-(f(a)+g(a))| &\le |f(x)-f(a)| + |g(x)-g(a)| \\ &< \varepsilon_1 + \varepsilon_2 \end{aligned}$$

At this point we decide what choice of ε_1 and ε_2 will make $\varepsilon_1 + \varepsilon_2 \le \varepsilon$.

After deciding that

$$\varepsilon_1 + \varepsilon_2 \le \varepsilon$$

when $\varepsilon_1 = \varepsilon/2$ and $\varepsilon_2 = \varepsilon/2$ we are done and can fill in the blanks left at the beginning of the proof.

In a Calculus book reference to ε_1 and ε_2 are often left out completely and the proof looks like

Let $\varepsilon > 0$. Since $\lim_{x \to a} f(x) = f(a)$ then there is $\delta_1 > 0$ such that $x \in D$ and $0 < |x-a| < \delta_1$ implies $|f(x) - f(a)| < \varepsilon/2$. Since $\lim_{x \to a} g(x) = g(a)$ then there is $\delta_2 > 0$ such that $x \in D$ and $0 < |x-a| < \delta_2$ implies $|g(x) - g(a)| < \varepsilon/2$.

Choose $\delta = \min(\delta_1, \delta_2)$. Then $x \in D$ and $0 < |x-a| < \delta$ implies

$$|f(x) + g(x) - (f(a) + g(a))| \le |f(x) - f(a)| + |g(x) - g(a)|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Note that when ε_1 and ε_2 are removed the inequality

$$\varepsilon_1 + \varepsilon_2 \le \varepsilon$$

doesnt appear anymore. Thus it is difficult to see what inequality was used to find ε_1 and ε_2 so that the proof works.

Rather than trying to remember that magic value for ε_1 and ε_2 that makes the textbook proof work out put ε_1 and ε_2 into the estimates and then solve the inequality to find ε_1 and ε_2.

Let's work the inequality for $\lim_{x \to a} f(x)g(x) = f(a)g(a)$.

The first part of the proof is the same, except we will be filling in the blanks

$$\varepsilon_1 = \boxed{} \quad \text{and} \quad \varepsilon_2 = \boxed{}$$

with different things. Thus

$$|f(x)g(x) - f(a)g(a)| = |f(x)g(x) - f(a)g(x)| + |f(a)g(x) - f(a)g(a)|$$
$$= |g(x)||f(x) - f(a)| + |f(a)||g(x) - g(a)|$$
$$< |g(x)|\,\varepsilon_1 + |f(a)|\,\varepsilon_2$$
$$\le (|g(x) - g(a)| + |g(a)|)\,\varepsilon_1 + |f(a)|\,\varepsilon_2$$
$$\le (\varepsilon_2 + |g(a)|)\,\varepsilon_1 + |f(a)|\,\varepsilon_2.$$

Now we need to choose values of ε_1 and ε_2 that will make

$$(\varepsilon_2 + |g(a)|)\,\varepsilon_1 + |f(a)|\,\varepsilon_2 \le \varepsilon.$$

There are many choices of ε_1 and ε_2 that can be chosen to satisfy this inequality. We'll pick a different choice than yesterday.

To simplify let $\varepsilon_2 = \varepsilon_1$ then we have

$$(\varepsilon_1 + |g(a)|)\varepsilon_1 + |f(a)|\varepsilon_1 = (\varepsilon_1 + M)\varepsilon_1 \le \varepsilon$$

where $M = |f(a)| + |g(a)|$.

Now solve for ε_1 so that $(\varepsilon_1 + M)\varepsilon_1 = \varepsilon$. This is a quadratic equation

$$\varepsilon_1^2 + M\varepsilon_1 - \varepsilon = 0$$

Thus

$$\varepsilon_1 = \frac{-M + \sqrt{M^2 + 4\varepsilon}}{2} = \frac{-|f(a)| - |g(a)| + \sqrt{(|f(a)| + |g(a)|)^2 + 4\varepsilon}}{2}$$

If we started the proof with

since $\lim_{x \to a} f(x) = f(a)$ then there exists $\delta_1 > 0$ such that $x \in D$ and $0 < |x-a| < \delta_1$ implies

$$|f(x) - f(a)| < \frac{-|f(a)| - |g(a)| + \sqrt{(|f(a)| + |g(a)|)^2 + 4\varepsilon}}{2}.$$

The inequality $(\varepsilon_1 + M)\varepsilon_1 \le \varepsilon$ would never appear in the proof and the means by which we chose ε_1 would not be clear. Moreover the proof in this form would be nearly impossible to remember.

2.89. Construct a sequence of continuous functions on $[0,1]$ that converge pointwise to a continuous function but for which the limit and integral cannot be interchanges.

If you can't find an example look back to the notes from the first day of class.

That example

$$f_n(x) = n|x|e^{-nx^2}$$

$$\int_{-\infty}^{\infty} f_n(x)dx = 1 \quad \text{so} \quad \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx = 1$$

$$\lim_{n\to\infty} f_n(x) = 0 \quad \text{so} \quad \int_{-\infty}^{\infty} \lim_{n\to\infty} f(x)\,dx = 0.$$

Does this example work for $[0,1]$?

$$\int_0^1 f_n(x)dx = \tfrac{1}{2}(1-e^{-n}) \to \tfrac{1}{2} \text{ as } n\to\infty.$$

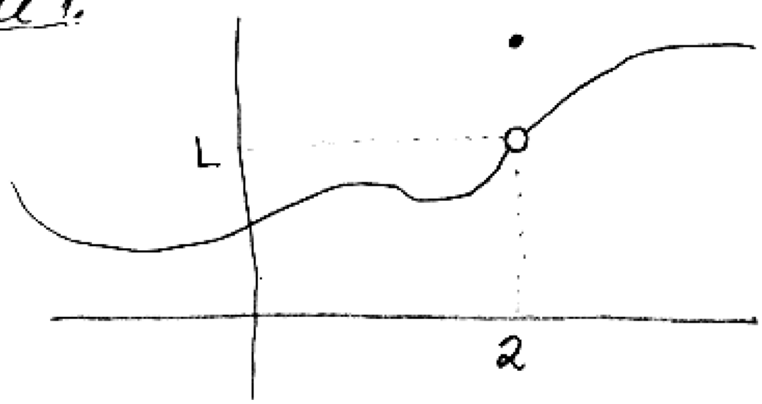
So yes. This same example works for $[0,1]$ and provides an answer to 2.89 in McDonald and Weiss.

Discussion of the extra credit problem

Let $f:[a,b]\to\mathbb{R}$ and $B=\{c: \lim_{x\to c} f(x)=L$ exists but $L\neq f(c)\}$.

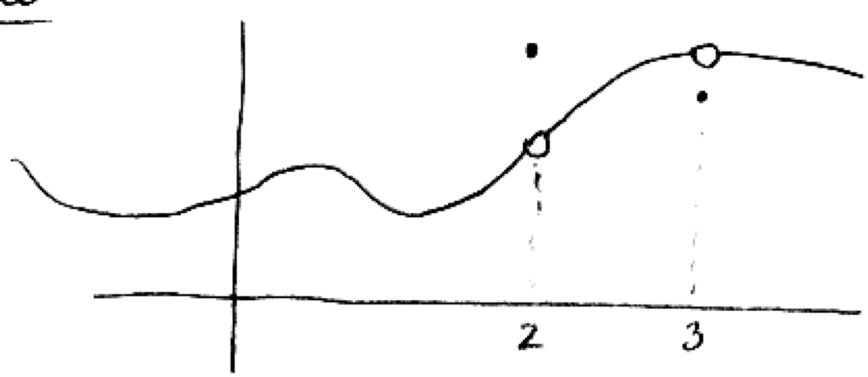
Prove or disprove B is a countable set.

Example 1.



In this case $B=\{2\}$ therefore B is countable.

Example 2



In this case $B=\{2,3\}$ therefore B is countable.

Example 3

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

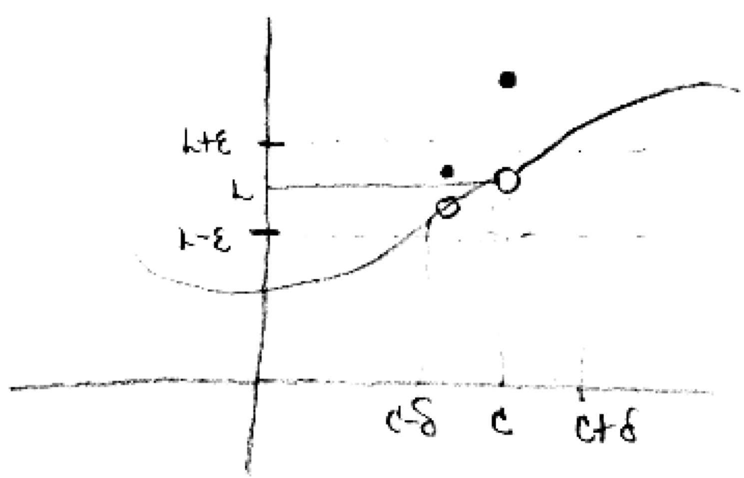
Recall that f is continuous on $\mathbb{R} \setminus \mathbb{Q}$ and discontinuous on $\mathbb{Q}$. Moreover since

$$\lim_{x \to a} f(x) = 0$$

for any $a \in \mathbb{R}$. Then each of the discontinuities are removable. Therefore $B = \mathbb{Q}$ is countable.

Is B in general countable? Perhaps we can use the Lindelöf theorem to prove this.

Let $c \in B$ then $\lim_{x \to c} f(x) = L$ which means for $\varepsilon > 0$ there is $\delta > 0$ such that $x \in [a, b]$ and $0 < |x - c| < \delta$ implies $|f(x) - L| < \varepsilon$.



So in the interval $(c-\delta, c+\delta)$ if there is any other removable discontinuity $c_2 \in (c-\delta, c+\delta) \cap B$. Then

$$\left|f(c_2) - \lim_{x \to c_2} f(x)\right| \le \left|f(c_2) - L\right| + \left|L - \lim_{x \to c_2} f(x)\right| \le 2\varepsilon.$$

This appears to provide some control on the size of the difference between the values of the function and where the hole in the graph is.

Consider $\varepsilon_n = \frac{1}{n}$ and defining

$$B_n = \left\{ c \in B : \left|f(c) - \lim_{x \to c} f(x)\right| \geq \varepsilon_n \right\}$$

Then

$$B = \bigcup_{n=1}^{\infty} B_n$$

If we can show each B_n is countable, then B would be the countable union of countable sets and therefore countable.

If you can show B_n is countable then you have proved the result. If B_n is not necessarily countable you may want to start looking for a counterexample.

A example of E where $\overline{E} \setminus E'$ is closed

Let $E = (0,1) \cup \{14\}$.

Then $\overline{E} = [0,1] \cup \{14\}$.

and $E' = [0,1]$.

Therefore $\overline{E} \setminus E' = \{14\}$ which for this example is a closed set.

Some observations

$\overline{E} \setminus E'$ being closed means $\overline{\overline{E} \setminus E'} = \overline{E} \setminus E'$.

Suppose $x \in \overline{\overline{E} \setminus E'}$. Then there is $x_n \in \overline{E} \setminus E'$ such that $x_n \to x$. Since $x_n \in \overline{E}$ then $x \in \overline{E}$.

If we can show $x \in \overline{E} \setminus E'$ then $\overline{E} \setminus E'$ is closed if we can't show this, then we might want to look for a counter example where $\overline{E} \setminus E'$ is not closed.

Suppose $x \in \overline{E} \setminus E'$. Then since $x \in \overline{E}$ then it must follow that $x \in E'$. If this leads to a contradiction then we have shown $\overline{E} \setminus E'$ is closed, otherwise we might want to continue looking for a counterexample.

Try working exercise 2.41 to get more familiar with the deffinition of accumulation point.

If $x \in \mathbb{R}$ is an accumulation point then there exists a sequence $x_n \in E$ of distinct elements where $x_n \to x$.

Since x is an accumulation point, then for each $\varepsilon > 0$ there is $y \in E$ such that $0 < |x-y| < \varepsilon$.

Choose $\varepsilon_n = 1/n$. Then there is $y_n \in E$ such that $0 < |x-y_n| < \varepsilon_n$.

Clearly $y_n \to x$ as $n \to \infty$, however y_n may not be a sequence of distinct elements.

Let $W = \{y_n ; n \in \mathbb{N}\}$. Claim W is infinite. If not then W would be finite and consequently

$$V = \{|x-y_n| : n \in \mathbb{N}\}$$

would also be finite. Since finite sets have a minimum then for some $N \in \mathbb{N}$ we have

$$\min V = |x-y_N| = \alpha > 0$$

But this contradicts $y_n \to x$. Therefore W is infinite. If we extract a subsequence y_{n_k} of distinct elements by induction. Then

$$y_{n_k} \to x \quad \text{as } k \to \infty$$

and y_{n_k} is a sequence of distinct elements.

Comments about maximum value of a finite sequence and product of two finite sequences

$$X = (1, 3, 5, 4)$$

$$Y = (4, 2, 3, 7)$$

Then $\max X_n = 5$, $\max Y_n = 7$

$$\max X_n Y_n = \max\{4, 6, 5, 28\} = 28.$$

Thus $\max X_n Y_n \leq (\max X_n)(\max Y_n)$.

On the other hand. If

$$X = (-1, -3, -5, -4)$$

$$Y = (-4, -2, -3, -7)$$

Then $\max X_n = -1$, $\max Y_n = -2$

$$\max X_n Y_n = 28$$

In this example

$$\max X_n Y_n \geq (\max X_n)(\max Y_n)$$

This may help with problem #1 on homework #3.