713 Review Summary

October 9

Let $S = \{x : x_n \to x$ for some sequence $x_n \in \mathbb{Q}\}$.

$T = \{y : y_n \to y$ for some sequence $y_n \in S\}$.

Question: Is $S = T$? YES

By definition $S = \overline{\mathbb{Q}} = \mathbb{R}$ and so $T = \overline{\mathbb{R}} = \mathbb{R}$.

Let $C = \{f : \mathbb{R} \to \mathbb{R} : f$ is continuous$\}$.

$S = \{f : f_n \to f$ pointwise for some sequence $f_n \in C\}$

$T = \{g : g_n \to g$ pointwise for some sequence $g_n \in S\}$

Question: Is $S = T$?

This question should be thought about as we start reading chapter 3 in McDonald and Weiss.

In particular if $\hat{C}$ is the smallest set of functions that contain the continuous functions and is closed under pointwise limits what is the relation between $\hat{C}$, S and T?

Define $\tau = \{ O \subseteq \mathbb{R} : O \text{ is open} \}$ and $\mathcal{B}$ to be the smallest σ-algebra containing τ. Then

$$C = \{ f : \mathbb{R} \to \mathbb{R} \text{ s.t. } f^{-1}(O) \in \tau \text{ for every } O \in \tau \}$$

$$\hat{C} = \{ f : \mathbb{R} \to \mathbb{R} \text{ st. } f^{-1}(O) \in \mathcal{B} \text{ for every } O \in \tau \}.$$

This is an alternative characterization of $\hat{C}$ which will be shown is section 3.1.

We worked problem 15 (iii) from Quiz 1 whose solution appears in the solution guide and will not be reproduced here.

<u>Question</u>: Let $f_n(x) = (1 + \frac{x}{n})^n$ and $f(x) = e^x$.

(1) Does $f_n \to f$ pointwise

(2) Does $f_n \to f$ uniformly.

<u>Answers</u>:

(1) Yes

(2) No.

<u>Discussion of (1)</u>: By the binomial theorem

$$f_n(x) = \left(1+\frac{x}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k}\left(\frac{x}{n}\right)^k = \sum_{k=0}^{n} \frac{n!}{(n-k)!\,k!}\left(\frac{x}{n}\right)^k$$

$$= \sum_{k=0}^{n} \frac{n\cdot(n-1)\cdots(n-k+1)}{n\cdot n\cdots\cdot n}\cdot\frac{1}{k!}\,x^k$$

$$= \sum_{k=0}^{n} 1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\frac{1}{k!}\,x^k$$

$$= \sum_{k=0}^{n} \alpha_{k,n}\,\frac{1}{k!}\,x^k$$

where $\alpha_{k,n} = \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$.

Note that $0 \leq \alpha_{k,n} \leq 1$ for all $0 \leq k \leq n$, and for k fixed

$$\lim_{n\to\infty} a_{k,n} = 0 .$$

By the Taylor's theorem

$$f(x) = e^x = \sum_{k=1}^{\infty} \frac{1}{k!}\,x^k .$$

Since $a_{k,n} \to 0$ as $n \to \infty$ it appears that we should be able to compare the sequences to show that $f_n \to f$ pointwise.

Given $n \geq N$ the comparison should go like

$$|f_n(x)-f(x)| \leq \left|f_n(x) - \sum_{k=0}^{N} \binom{n}{k}\left(\frac{x}{n}\right)^k\right|$$

$$+\left|\sum_{k=0}^{N}\binom{n}{k}\left(\frac{x}{n}\right)^k - \sum_{k=0}^{N}\frac{1}{k!}x^k\right|$$

$$+\left|\sum_{k=0}^{N}\frac{1}{k!}x^k - f(x)\right|$$

It appears to be an $\varepsilon/3$ argument. Let's look at each term before trying to choose N and n. Hence for $n \geq N$ we have

$$\left|f_n(x) - \sum_{k=0}^{N}\binom{n}{k}\left(\frac{x}{n}\right)\right| = \left|\sum_{k=N+1}^{n}\binom{n}{k}\left(\frac{x}{n}\right)\right| = \left|\sum_{k=N+1}^{n}\alpha_{k,n}\frac{1}{k!}x^k\right|$$

$$\leq \sum_{k=N+1}^{n}\alpha_{k,n}\frac{1}{k!}|x|^k \leq \sum_{k=N+1}^{n}\frac{1}{k!}|x|^k \leq \sum_{k=N+1}^{\infty}\frac{1}{k!}|x|^k$$

Since

$$\sum_{k=0}^{\infty}\frac{1}{k!}|x|^k = e^{|x|} < \infty$$

is convergent then this term can be made small by choosing N large enough.

Clearly the same value of N gives good bounds on the term

$$\left|\sum_{k=0}^{N} \frac{1}{k!}x^k - f(x)\right| = \left|\sum_{k=N+1}^{\infty} \frac{1}{k!}x^k\right| \leq \sum_{k=N+1}^{\infty} \frac{1}{k!}|x|^k .$$

What's left is to choose n large enough so that $n \geq N$ and

$$\left|\sum_{k=0}^{N} \binom{n}{k}\left(\frac{x}{n}\right)^k - \sum_{k=0}^{N} \frac{1}{k!}x^k\right| = \sum_{k=0}^{N} (1-\alpha_{k,n})\frac{1}{k!}|x|^k$$

is small. This can be done since $\alpha_{k,n} \to 1$ as $n \to \infty$.

Now let's put everything together into the proof.

<u>Proof of (1)</u>: Let $x \in \mathbb{R}$. If $x=0$ then $f_n(0)=1$ and $f(0)=1$ so clearly $f_n(0) \to f(0)$ as $n \to \infty$. Otherwise, let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ so large that

$$\sum_{k=N+1}^{\infty} \frac{1}{k!}|x|^k < \varepsilon/3 .$$

Since $\alpha_{k,n} \to 1$ as $n \to \infty$ then for each $k = 1, 2, \ldots, N$ choose $n_k \geq N$ such that $n \geq n_k$ implies

$$|1-\alpha_{k,n}| < \frac{k!\,\varepsilon}{3(N+1)|x|^k} .$$

Let $M = \max\{n_k : k=0,1,\dots,N\}$. Then $M \geq n_k$ for each $k=0,1,\dots,N$ and therefore $n \geq M$ implies

$$\sum_{k=0}^{N}(1-\alpha_{k,n})\frac{1}{k!}|x|^k \leq \sum_{k=0}^{N}\frac{k!\,\varepsilon}{3(N+1)|x|^k}\frac{1}{k!}|x|^k = \varepsilon/3$$

It follows that for $n \geq M$ we have that $n \geq N$ and

$$|f_n(x)-f(x)| \leq \left|f_n(x)-\sum_{k=0}^{N}\binom{n}{k}\left(\frac{x}{n}\right)^k\right|$$

$$+\left|\sum_{k=0}^{N}\binom{n}{k}\left(\frac{x}{n}\right)^k-\sum_{k=0}^{N}\frac{1}{k!}x^k\right|$$

$$+\left|\sum_{k=0}^{N}\frac{1}{k!}x^k-f(x)\right|$$

$$< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$$

which shows that $f_n \to f$ pointwise as $n \to \infty$.

There is a simpler proof based on logarithms that typically done in Calculus class.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty} e^{n\log(1+\frac{x}{n})} = \lim_{h\to 0^+} e^{\left(\frac{\log(1+hx)}{h}\right)}$$

$$= e^{\lim_{h\to 0^+}\left(\frac{\log(1+hx)}{h}\right)} = e^{\lim_{h\to 0}\left(\frac{\frac{1}{1+hx}\cdot x}{1}\right)} = e^x = f(x).$$

Discussion of (2): We want to show that $f_n \to f_n$ uniformly is false. That is, that

$$\text{not } (f_n \to f \text{ uniformly})$$

is true.

Recall the definition of uniform convergence.

Let $f_n : D \to \mathbb{R}$. Then $f_n \to f$ uniformly means

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $n \geq N$ and $x \in D$ implies $|f_n(x) - f(x)| < \varepsilon$.

Therefore not ($f_n \to f$ uniformly) means

$\exists \varepsilon > 0$ st $\forall N \in \mathbb{N}$ $\exists n \geq N$ and $\exists x \in D$ s.t. $|f_n(x) - f(x)| \geq \varepsilon$.

I get to choose ε

you choose N

then I get to choose $n \geq N$ and $x \in D$

such that $|f_n(x) - f(x)| \geq \varepsilon$.

So let's work this inequality.

For example if we choose $\varepsilon = 1/2$ then given $N \in \mathbb{N}$ we could solve the inequality

Find all $n \geq N$ and $x \in \mathbb{R}$ such that

$$\left| \left(1+\frac{x}{n}\right)^n - e^x \right| \geq 1/2 .$$

Actually we don't have to find all solutions to this inequality, but only one choice of $n \geq N$ and $x \in \mathbb{R}$ that satisfy it.

Ideas?

How about $n = N$ and $x = n^2$?

Does it work?

This would mean

$$\left| (1+n)^n - e^{n^2} \right| \geq 1/2$$

for all $n \in \mathbb{N}$. Is it true? How try to prove by induction.

Clearly if $n=1$ then

$$|(1+1)^1 - e^1| \approx .7 > \frac{1}{2}.$$

Suppose, for induction, that

$$|(1+n)^n - e^{n^2}| \geq \frac{1}{2}$$

Claim that

$$|(1+n+1)^{n+1} - e^{(n+1)^2}| \geq \frac{1}{2}.$$

To make easier how about try to prove slightly simpler result.

$$e^{n^2} - (1+n)^n \geq \frac{1}{2}$$

implies

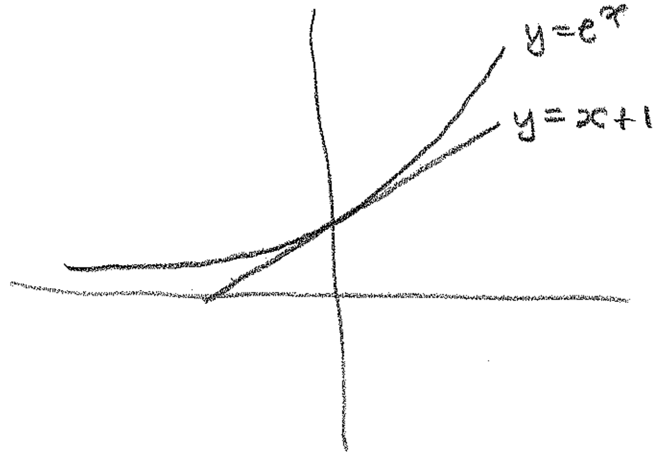
$$e^{(n+1)^2} - (1+n+1)^{n+1} \geq \frac{1}{2}$$

Estimate

$$e^{(n+1)^2} = e^{n^2} e^{n+1} \geq \left(\frac{1}{2} + (1+n)^n\right) e^{n+1}$$

Now what?

What about the inequality



which says $e^x \geq x+1$ for all x. Then

$$e^{n+1} \geq n+2 = 1+n+1$$

and

$$e^{(n+1)^2} \geq \left(\tfrac{1}{2} + (1+n)^n\right)(1+(n+1))$$

$$= \tfrac{1}{2} + (1+n)^n + \frac{n+1}{2} + (1+n)^{n+1}$$

Stuck. We want this to be greater than

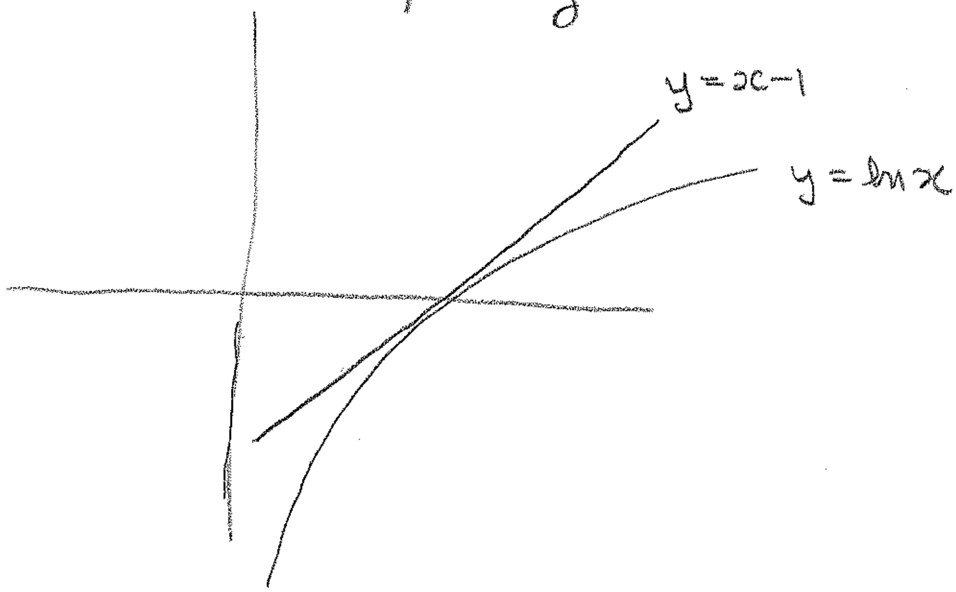
$$\tfrac{1}{2} + (1+n+1)^{n+1}.$$

but this doesn't seem likely. Let's try working on the other side.

Thus

$$(1+n+1)^{n+1} = e^{(n+1)\ln(1+n+1)}$$

and since the inequality



Say $\ln x \le x-1$ for all $x>0$. Then

$$\ln(1+n+1) \le n+1$$

we obtain that

$$(1+n+1)^{n+1} \le e^{(n+1)^2}$$

which is missing the $\frac{1}{2}$ so still not quite close enough.

One option it to try better estimates; another is to choose a different value for x. Alternatively we can try to make a proof that doesn't depend on an explicit choice of x and therefore doesn't require such carefull estimates.

Recall for any polynomial $p(x)$ that

$$\lim_{x \to \infty} \frac{p(x)}{e^x} = 0.$$

Thus

$$\lim_{x \to \infty} \frac{\left(1 + \frac{x}{n}\right)^n}{e^x} = 0.$$

So for $\varepsilon_2 > 0$ there is $x > 0$ large enough such that

$$\left| \frac{\left(1 + \frac{x}{n}\right)^n}{e^x} \right| < \varepsilon_2$$

We will choose a particular value of ε_2 in light of what is needed to make the estimate work out.

Estimate

$$e^x - \left(1+\frac{x}{n}\right)^n = e^x\left(1 - \frac{\left(1-\frac{x}{n}\right)^n}{e^x}\right)$$

$$\geq e^x(1-\varepsilon_2)$$

So we need $\varepsilon_2 < 1$ to make the right side positive. If this is the case we can procede by estimating

$$x > 0$$

$$e^x > e^0 = 1$$

and so

$$e^x - \left(1+\frac{x}{n}\right)^n \geq 1-\varepsilon_2$$

Now taking $\varepsilon_2 = \frac{1}{2}$ we obtain that

$$\left|e^x - \left(1+\frac{x}{n}\right)^n\right| \geq \frac{1}{2}$$

Note that we did not find x explicitly in terms of n but simply used the limit $\lim_{x\to\infty} \frac{(1+x/n)^n}{e^x} = 0$ to show x existed.

Sometimes an explicit estimate for x in terms of n is useful for applications. The proof that finds such an estimate for x is typically called "hard" analysis, where the proof we just finished is considered to be "soft" or "softer."

Sometimes the hypothesis are so general that this "softer" proof is the only option.

A few questions:

(1) Why does $\sum_{k=0}^{\infty} \frac{1}{k!}|x|^k$ converge.

(2) Why is $\lim_{x \to \infty} \frac{P(x)}{e^x} = 0$ for any polynomial $P(x)$.

Discussion of (1).

Does anyone remember the ratio test from Calculus class?

<u>Ratio Test</u>: If $a_k > 0$ and $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$ then $\sum_{k=0}^{\infty} a_k < \infty$.

The proof is by comparison with the geometric series $\sum_{k=0}^{\infty} \alpha^k$

which converges for positive α exactly when $\alpha < 1$.

<u>Discussion of (2)</u>:

Let n be the degree of $P(x)$. Then

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

By L'Hôpitals rule

$$\lim_{x\to\infty} \frac{P(x)}{e^x} = \lim_{x\to\infty} \frac{n!\, a_n}{e^x} = 0.$$

Since $\left(1+\frac{x}{n}\right)^n$ is a polynomial of degree n

then $$\lim_{x\to\infty} \frac{\left(1+\frac{x}{n}\right)^n}{e^x} = 0.$$