Math 713 Summary: Oct 14, '10

Last time we defined

$C = \{f : \mathbb{R} \to \mathbb{R}$ s.t. f is continuous$\}$

$\hat{C}$ = smallest collection of functions that is closed under pointwise limits and contains the continuous functions.

and proved the following facts:

1. $\hat{C}$ is an algebra of functions
2. $\chi_O \in \hat{C}$ for every open set $O \subseteq \mathbb{R}$.
3. If $f \in \hat{C}$ then $|f| \in \hat{C}$.
4. If $f_n \in \hat{C}$ then $\sup f_n \in \hat{C}$ and $\inf f_n \in \hat{C}$.

At the end of the lecture we defined

$\mathcal{T} = \{O \subseteq \mathbb{R} : O \text{ is open}\}$.

$\mathcal{B} = \{A \in \mathbb{R} : \chi_A \in \hat{C}\}$.

We finish the chapter upon proving two claims:

<u>Claim</u>: $\mathcal{B} = \mathcal{A}(\mathcal{T})$ is the smallest σ-algebra that contains the open sets

<u>Claim</u>: $\hat{C} = \{f : \mathbb{R} \to \mathbb{R}$ s.t. $f^{-1}(O) \in \mathcal{B}$ for $O \in \mathcal{T}\}$.

We establish a few more facts before proving these claims.

First we prove the following subclaims:

Subclaim $\mathcal{B}$ contains the open sets, and

Subclaim $\mathcal{B}$ is a σ-algebra.

The first subclaim follows from fact (2) proved last time. To prove the second subclaim we need to show that

(i) $\mathcal{B}$ is closed under complements.

(ii) $\mathcal{B}$ is closed under countable unions.

Proof of (i). Let $A \in \mathcal{B}$. Then $\chi_A \in \hat{C}$.

Since $\hat{C}$ is an algebra then $1 - \chi_A \in \hat{C}$

Since $1 - \chi_A = \chi_{A^c}$ then $\chi_{A^c} \in \hat{C}$.

Therefore $A^c \in \mathcal{B}$.

Proof of (ii) Let $A_n \in \mathcal{B}$. Then $\chi_{A_n} \in \hat{C}$ for $n \in \mathbb{N}$.

By fact (4) from yesterday $\sup \chi_{A_n} \in \hat{C}$.

Since $\sup \chi_{A_n} = \chi_{\cup A_n}$ then $\chi_{\cup A_n} \in \hat{C}$.

Therefore $\cup A_n \in \hat{C}$.

We have now shown that $\mathcal{B}$ is a σ-algebra.

Now we take a break and think about the following which could have appeared on the quiz.

Let $\mathcal{B}$ be a σ-algebra of subsets of $\mathbb{R}$ and $f: \mathbb{R} \to \mathbb{R}$ be any function. Define $\mathcal{A} = \{A \subseteq \mathbb{R} : f^{-1}(A) \in \mathcal{B}\}$. Prove or disprove the claim that $\mathcal{A}$ is a σ-algebra.

After the break will have a vote whether to prove or disprove this claim.

<u>Some propaganda before the vote.</u>

Is $f(x) = 1$ a counterexample?

<u>Vote</u>:

Claim is true	There exists a counter example	Total
16	2	18

<u>Conclusion</u>: Some people did not vote.

Let's try the suggestion for a counter example. First let's explicity find $f^{-1}(A)$ when $f(x) = 1$.

$$f^{-1}(A) = \begin{cases} \mathbb{R} & \text{if } 1 \in A \\ \emptyset & \text{if } 1 \notin A \end{cases}$$

Now if $\mathcal{B}$ is any σ-algebra of subsets of $\mathbb{R}$. do we know that $\emptyset \in \mathcal{B}$?

Since $\mathcal{B}$ is a σ-algebra it is nonempty, therefore there is $A \in \mathcal{B}$.

Since σ-algebras are closed under complement then $A^c \in \mathcal{B}$.

Since σ-algebras are closed under union then $A \cup A^c = \mathbb{R} \in \mathcal{B}$.

Since σ-algebras are closed under complement then $\mathbb{R}^c = \emptyset \in \mathcal{B}$.

Therefore $\emptyset \in \mathcal{B}$ and also $\mathbb{R} \in \mathcal{B}$.

What is $\mathcal{A} = \{A \subseteq \mathbb{R} : f^{-1}(A) \in \mathcal{B}\}$ when $f(x) = 1$?

Since we have just shown that $\emptyset, \mathbb{R} \in \mathcal{B}$ then $f^{-1}(A) \in \mathcal{B}$ for and $A \subseteq \mathbb{R}$.

Therefore $\mathcal{A} = \mathcal{P}(\mathbb{R})$ = power set of $\mathbb{R}$.

Is $\mathcal{A}$ a σ-algebra? YES

Since the counter example failed, lets try to prove the claim. Thus we need to show

(i) $\mathcal{A}$ is closed under complements

(ii) $\mathcal{A}$ is closed under countable unions.

<u>Proof of (i)</u> Let $A \in \mathcal{A}$. Then $f^{-1}(A) \in \mathcal{B}$.

Denote $B = f^{-1}(A)$.

Since $\mathcal{B}$ is closed under complements then $B^c \in \mathcal{B}$

Then $B^c = f^{-1}(A)^c = f^{-1}(A^c) \in \mathcal{B}$ implies $A^c \in \mathcal{A}$.

<u>Proof of (ii)</u> Let $A_n \in \mathcal{A}$. Then $f^{-1}(A_n) \in \mathcal{B}$.

Denote $B_n = f^{-1}(A_n)$ for $n \in \mathbb{N}$

Since $\mathcal{B}$ is closed under countable union then $\bigcup_{n=1}^{\infty} B_n \in \mathcal{B}$.

Then $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} f^{-1}(A_n) = f^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) \in \mathcal{B}$ implies $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

We have now shown that $\mathcal{A}$ is a σ-algebra.

Our goal is to show $\mathcal{B}$ is the smallest σ-algebra containing $\mathcal{T}$ and that $\hat{C} = \mathcal{F}$ where

$$\mathcal{F} = \{ f : \mathbb{R} \to \mathbb{R} \text{ st } f^{-1}(O) \in \mathcal{B} \text{ for } O \in \mathcal{T} \}.$$

To achieve this goal we give another characterization of $\mathcal{F}$ that will be useful.

Define

$$\mathcal{F} = \{ f : \mathbb{R} \to \mathbb{R} \text{ s.t. } f^{-1}(O) \in \mathcal{B} \text{ for } O \in \tau \}$$

$$\mathcal{F}_1 = \{ f : \mathbb{R} \to \mathbb{R} \text{ st } f^{-1}((-\infty, a)) \in \mathcal{B} \text{ for } a \in \mathbb{R} \}.$$

$$\mathcal{F}_2 = \{ f : \mathbb{R} \to \mathbb{R} \text{ st. } f^{-1}((a, \infty)) \in \mathcal{B} \text{ for } a \in \mathbb{R} \}.$$

<u>Claim</u> $\mathcal{F} = \mathcal{F}_1 = \mathcal{F}_2$.

Obviously ? $\mathcal{F} \subseteq \mathcal{F}_1$ and $\mathcal{F} \subseteq \mathcal{F}_2$.

If there are no questions we can remove the "?" after "obviously" and continue.

<u>Claim</u> $\mathcal{F}_1 \subseteq \mathcal{F}$. Let $f \in \mathcal{F}_1$ and $\mathcal{A}$ the given by

$$\mathcal{A} = \{ A \subseteq \mathbb{R} : f^{-1}(A) \in \mathcal{B} \}.$$

By the previous result $\mathcal{A}$ is a σ-algebra. By the definition of $\mathcal{F}_1$ we have $(-\infty, a) \in \mathcal{A}$ for every $a \in \mathbb{R}$. Let $a = \alpha + \frac{1}{n}$. Then $(-\infty, \alpha + \frac{1}{n}) \in \mathcal{A}$ and $\mathcal{A}$ a σ-algebra implies that $(-\infty, \alpha] = \bigcap_{n=1}^{\infty} (-\infty, \alpha + \frac{1}{n}) \in \mathcal{A}$ since $\mathcal{A}$ is closed under countable intersections.

Now $(-\infty,\alpha] \in \mathcal{A}$ and $\mathcal{A}$ closed under complement implies $(\alpha,\infty) \in \mathcal{A}$. Therefore

$$\{(-\infty,a): a \in \mathbb{R}\} \cup \{(\alpha,\infty): \alpha \in \mathbb{R}\} \subseteq \mathcal{A}.$$

It follows that

$$(a,b) = (-\infty,b) \cap (a,\infty) \in \mathcal{A}.$$

In particular $\mathcal{A}$ contains the open intervals. Now, let $O \subseteq \mathbb{R}$ be an arbitrary open set. By the structure theorem of the open sets of $\mathbb{R}$ we have that $O = \bigcup (a_n, b_n) = \bigcup I_n$ is a countable disjoint union of open intervals. Since $\mathcal{A}$ is a σ-algebra $O \in \mathcal{A}$. By definiton of $\mathcal{A}$ it follows that $f^{-1}(O) \in \mathcal{B}$. Therefore $f \in \mathcal{F}$.

Therefore $\mathcal{F} = \mathcal{F}_1$. Similarly $\mathcal{F} = \mathcal{F}_2$.

We want to show $\hat{C} = \mathcal{F}$. Since $\hat{C}$ contains the continuous functions and is closed under pointwise limits we should be able to show

(i) $\mathcal{F}$ contains the continuous functions

(ii) $\mathcal{F}$ is closed under pointwise limits.

One of these is obvious. Which one?

Why?

$$\mathcal{F} = \{ f : \mathbb{R} \to \mathbb{R} \text{ st } f^{-1}(O) \in \mathcal{B} \text{ for } O \in \tau \}$$

$$C = \{ f : \mathbb{R} \to \mathbb{R} \text{ st. } f^{-1}(O) \in \tau \text{ for } O \in \tau \}$$

Since $\mathcal{B}$ contains the open sets $\tau \subseteq \mathcal{B}$.

Since $\tau \subseteq \mathcal{B}$ then $C \subseteq \mathcal{F}$.

Therefore $\mathcal{F}$ contains the continuous functions.

<u>Before proving (ii) consider the claim</u>

Let $g_n : \mathbb{R} \to \mathbb{R}$ be a sequence of functions and define $g(x) = \sup\{ g_n(x) : n \in \mathbb{N} \}$. Prove or disprove that

$$g^{-1}((a,\infty)) = \bigcup_{n=1}^{\infty} g_n^{-1}((a,\infty)).$$

There is no time to vote. The claim is true. Can you see why?

Proof: We need to show

$$\{x : \sup\{g_n(x) : n \in \mathbb{N}\} > a\} = \bigcup_{n=1}^{\infty} \{x : g_n(x) > a\}.$$

"$\subseteq$". Suppose $g(x_0) = \sup\{g_n(x_0) : n \in \mathbb{N}\} > a$. Since $g(x_0)$ is the least upper bound and $a < g(x_0)$ then a is not an upper bound. It follows there is $N \in \mathbb{N}$ such that $a < g_N(x_0)$. Thus

$$x_0 \in \{x : g_N(x) > a\} \subseteq \bigcup_{n=1}^{\infty} \{x : g_n(x) > a\}.$$

"$\supseteq$". Suppose $x_0 \in \bigcup_{n=1}^{\infty} \{x : g_n(x) > a\}$. Then for some $N \in \mathbb{N}$ we have $x_0 \in \{x : g_N(x) > a\}$. In particular $g_N(x_0) > a$. Since

$$g(x_0) = \sup\{g_n(x_0) : n \in \mathbb{N}\} \geq g_N(x_0) > a$$

it follows that $x_0 \in \{x : \sup\{g_n(x) : n \in \mathbb{N}\} > a\}$.

We finish showing $\mathcal{F}$ is closed under pointwise limits next time. The idea is if $f_n \to f$ pointwise, then

$$\lim_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x) = \lim_{n\to\infty} \sup\{f_k(x) : k \geq n\}$$

$$= \inf\left\{ \sup\{f_k(x) : k \geq n\} : n \in \mathbb{N} \right\}$$

and that the previous result can be used to show that $\mathcal{F}$ is closed under countable infima and suprema.