Math 713 Review   Oct 16, 2010

Tell a little about the proof which shows

if $f \in \hat{C}$ then $|f| \in \hat{C}$.

If $p_n(x) \to |x|$ why does $f_n(x) \to |f(x)|$?

Recall

$$f_n(x) = p_n(f(x)).$$

To see $f_n(x) \to |f(x)|$ we use the definition of pointwise convergence.

Given $x \in \mathbb{R}$ and $\varepsilon > 0$ define $y = f(x)$.

Since $p_n(y) \to |y|$ as $n \to \infty$ then there is $N \in \mathbb{N}$ large enough so that $n \geq N$ implies $|p_n(y) - |y|| < \varepsilon$.

Thus for $n \geq N$ we have

$$\big| f_n(x) - |f(x)| \big| = \big| p_n(f(x)) - |f(x)| \big|$$

$$= \big| p_n(y) - |y| \big| < \varepsilon$$

which shows that $f_n(x) \to |f(x)|$ as $n \to \infty$

Is this a special property of polynomials?

Let $g_n \to g$ pointwise and $f: \mathbb{R} \to \mathbb{R}$ be any function. Prove or disprove $(g_n \circ f)(x) \to (g \circ f)(x)$ as $n \to \infty$.

The claim is true. The proof is the same as the previous proof. Namely,

Given $x \in \mathbb{R}$ and $\varepsilon > 0$ define $y = f(x)$. Since $g_n(y) \to g(y)$ as $n \to \infty$ then there is $N \in \mathbb{N}$ large enough so that $n \geq N$ implies $|g_n(y) - g(y)| < \varepsilon$.

Thus for $n \geq N$ we have

$$|(g_n \circ f)(x) - (g \circ f)(x)| = |g_n(f(x)) - g(f(x))|$$
$$= |g_n(y) - g(y)| < \varepsilon$$

which shows that $(g_n \circ f)(x) \to (g \circ f)(x)$ as $n \to \infty$.

It may be possible to understand the previous proof more by considering the following claim:

Let $g_n \to g$ pointwise and $f: \mathbb{R} \to \mathbb{R}$ be any function. Prove or disprove $(f \circ g_n)(x) \to (f \circ g)(x)$ as $n \to \infty$.

Can you see the difference in this result?

The order of the composition is reversed.

Does it make a difference?

YES!

Now the convergence is essentially the definition of the continuity of f.

To find a counterexample one needs to create a suitable function

$$f: \mathbb{R} \to \mathbb{R}$$

which is not continuous.

Is the function $f(x) = \frac{1}{x}$ continuous?

YES!

The domain of $\frac{1}{x}$ is $(-\infty, 0) \cup (0, \infty)$ and this function is continuous at every point in its domain.

Is $\tan x$ continuous?

YES!

The domain of $\tan x$ is

$$\{x : \cos x \neq 0\} = \bigcup_{n \in \mathbb{Z}} \left(-\frac{\pi}{2} + n, \frac{\pi}{2} + n\right)$$

and this function is continuous at every point in its domain.

If $p(x)$ and $q(x)$ are polynomials then is $p(x)/q(x)$ continuous?

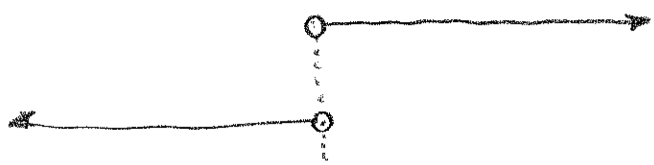
YES!

A rational function is continuous at every point in its domain.

Moreover, none of the functions on the previous page have domain $\mathbb{R}$ so they are not good as an example of a function $f : \mathbb{R} \to \mathbb{R}$.

Are there any discontinuous functions?

How about f defined by



$$F(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

is this continuous?

YES!

How about F defined by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

is this continuous?

No!

Do all discontinuous functions $f: \mathbb{R} \to \mathbb{R}$ have to be piecewise defined?

This question may not be precise enougth to have an answer, but seems plausible.

Why do all these functions "look" like they should be discontinuous?

$$\frac{1}{x}, \quad \tan x, \quad \frac{p(x)}{q(x)}.$$

Note that the domain is disconnected in each of these cases and this is how the functions look like they are made out of so many pieces.

Note that the function

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

can be made continuous by removing the point $x=0$ from its domain. to obtain

$$F(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Is it posible to make any function continuous by removing some points from its domain?

Is the function whose domain is $\emptyset$ continuous? Is it even a function?

To avoid removing too many points from the domain we could ask, given a function $f: \mathbb{R} \to \mathbb{R}$ how many points must be removed from the domain so that $F: D \to \mathbb{R}$ is continuous?

This is one of the questions we shall ask and answer about measurable functions. Note that χ_O is Borel measurable for any open set $O \subseteq \mathbb{R}$ and unless $O = \emptyset$ or $\mathbb{R}$ then χ_O is not continuous.

If we define $B = \overline{O} \setminus O$ and $D = \mathbb{R} \setminus B$. then is χ_O restricted to D continuous?

For an arbitrary function $f \in \hat{C}$ how many points must be removed so that

$$F = f|_D : D \to \mathbb{R}$$

is continuous. In other words,

Can we say something about how small $\mathbb{R}\setminus D$ is? How to measure the size of $\mathbb{R}\setminus D$?

(1) Is $\mathbb{R}\setminus D$ countable?

In the case χ_O then the structure theorem of the real numbers tells us that

$$O = \bigcup_n (a_n, b_n)$$

a countable union of open intervals. In this case $\mathbb{R}\setminus D = B = \overline{O}\setminus O = \{a_n : n \in \mathbb{N}\} \cup \{b_n : n \in \mathbb{N}\}$ which is countable.

Given any $f \in \hat{C}$ is there a D so that

$$F = f|_D : D \to \mathbb{R}$$

is continuous and $\mathbb{R}\setminus D$ countable?

If not, can we show $\lambda^*(\mathbb{R}\setminus D) = 0$?

Here is a counterexample to the claim that $(f \circ g_n)(x) \to (f \circ g)(x)$ as $n \to \infty$.

Let $f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$

and $g_n(x) = \frac{1}{n}$.

Then $\lim_{n\to\infty} (f \circ g_n)(x) = \lim_{n\to\infty} f\left(\frac{1}{n}\right)$

$= \lim_{n\to\infty} 0 = 0$

whereas $g_n(x) = \frac{1}{n} \to 0 = g(x)$ as $n \to \infty$ so

$$(f \circ g)(x) = f(0) = 1.$$

Therefore, in general, $(f \circ g_n)(x)$ does not converge pointwise to $(f \circ g)(x)$.

See if you can prove that if $f: \mathbb{R} \to \mathbb{R}$ is continuous that $g_n \to g$ pointwise implies $f \circ g_n \to f \circ g$ pointwise.

What is a polynomial?

This question is important to understand for the extra credit problem. We want the set $\mathbb{P}$ of all polynomials to be an algebra of functions. Thus

(1) If $p, q \in \mathbb{P}$ then $p+q \in \mathbb{P}$.

(2) If $p \in \mathbb{P}$ and $\alpha \in \mathbb{R}$ then $\alpha p \in \mathbb{R}$

(3) If $p, q \in \mathbb{R}$ then $pq \in \mathbb{P}$.

Obviously

$$p = x^2 + 14 \quad \text{and} \quad q = -x^2 + 8$$

are both polynomials. Since $p+q \in \mathbb{P}$ then

$$p+q = x^2+14-x^2+8 = 22 \in \mathbb{P}.$$

Let $\alpha \in \mathbb{R}$. Since $22 \in \mathbb{P}$ then

$$\alpha 22 \in \mathbb{P}$$

Therefore any constant function is is $\mathbb{P}$.
In particular, when $\alpha = 0$ we have

$$0 \in \mathbb{P}.$$

What is the degree of this polynomial?

Given a polynomial the degree is defined as the largest power of x appearing with a non-zero coefficient. Thus

$$\deg(x^2+14)=2$$

$$\deg(0x^5+x-1)=1$$

$$\deg(22x^0)=0$$

But $p(x)=0$ has no non-zero coefficients for any power of x.

Generally

$$\deg(p+q) \leq \max\{\deg p, \deg q\}$$

and

$$\deg(pq) = \deg p + \deg q.$$

Is it possible to define $\deg(0)$ so the above properties of degree hold?

So in some ways 0 is not a polynomial. However, if $\mathbb{P}$ is to be an algebra of functions it must contain 0.

For the extra credit problem lets assume that 0 is a polynomial. This avoids the trivial counter example

$$p_n = \frac{1}{n}$$

$$p_n \to 0 \quad \text{uniformly}$$

$$\deg p_n \leq 17 \quad \text{for all } n.$$

but 0 is not a polynomial.

Because, for this exercise, let's consider 0 to be a polynomial.

Note that in the proof of the Weierstrass approximation theorem given in the handout $p(x)=0$ is also considered a polynomial. In particular the approximating series of polynomials to the function $f(x)=0$ is $B_n(x)=0$ for all n.

Suppose $f_n \to f$ uniformly on (a,b) and f_n is differentiable for all n. Prove or disprove that $f_n' \to f'$.

Try to prove this using the fundamental theorem of calculus.

We want to show

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

exists, and we know

$$f_n'(x) = \lim_{h\to 0} \frac{f_n(x+h)-f_n(x)}{h}$$

exists. Also the fundemental theorem says that

$$f_n(x+h) - f_n(x) = \int_x^{x+h} f_n'(s)\,ds.$$

Now what? Try $\varepsilon/3$ argument.

Let's state the definition of uniform continuity.

$f_n \to f$ uniformly on (a,b) if

$\forall \varepsilon > 0\ \exists n \in \mathbb{N}$ s.t. $n \geq N$ and $x \in (a,b)$
implies $|f_n(x) - f(x)| < \varepsilon$.

Now the estimate ...

$$\frac{f(x+h)-f(x)}{h} = \frac{f(x+h)-f_n(x+h)}{h} + \frac{f_n(x+h)-f_n(x)}{h} + \frac{f_n(x)-f(x)}{h}.$$

We want to choose n so small that

$$\left|\frac{f(x+h)-f_n(x+h)}{h}\right| < \frac{\varepsilon}{3} \text{ and } \left|\frac{f_n(x)-f(x)}{h}\right| < \frac{\varepsilon}{3}$$

So n must depend on h.

We want to choose h so small that

$$\left|\frac{f_n(x+h)-f_n(x)}{h}\right| < \frac{\varepsilon}{3}$$

so h must depend on n. Oh dear, this is a circular set of dependencies. Can uniformity help us any more?

Let's just try to show that $f_n'(x)$ is pointwise cauchy. That should be easier.

By the fundamental theorem

$$\frac{f_n(x+h)-f_n(x)}{h} = \frac{1}{h}\int_x^{x+h} f_n'(s)\,ds.$$

Given $\varepsilon>0$ the cauchy condition means we want to find $N\in\mathbb{N}$ such that $n,m\geq N$ implies $|f_n'(x)-f_m'(x)|<\varepsilon$.

Again try an $\varepsilon/3$ argument...

$$|f_n'(x)-f_m'(x)| = \left|f_n'(x) - \frac{1}{h}\int_x^{x+h} f_n'(s)\,ds\right|$$

$$+\left|\frac{1}{h}\int_x^{x+h} f_n'(s)\,ds - \frac{1}{h}\int_x^{x+h} f_m'(s)\,ds\right|$$

$$+\left|\frac{1}{h}\int_x^{x+h} f_m'(s)\,ds - f_m'(x)\right|.$$

$$\leq \frac{1}{|h|}\left|\int_x^{x+h} |f_n'(x)-f_n'(s)|\,ds\right| + \frac{1}{|h|}\left|\int_x^{x+h} |f_n'(s)-f_m'(s)|\,ds\right| + \frac{1}{|h|}\left|\int_x^{x+h} |f_m'(s)-f_m'(x)|\,ds\right|$$

Now what?

It seems to need the continuity of f_n and f_m to go any farther.

Perhaps should start looking for a counter example.

It is time to take a break. Pehaps an idea will come while doing something else.

Try to look this theorem up on the internet or in your undergraduate analysis book to see if the theorem is stated correctly and the proof.