Math 713 Review Oct 16, '10

Last time we tried to prove:

<u>Claim</u>: If $f_n \to f$ uniformly on (a,b) and f_n is differentiable then $f_n' \to f'$.

and failed. Let's think about a counterexample.

How about $f_n(x) = \frac{\sin nx}{n}$

Then $f_n \to 0$ uniformly but $f_n'(x) = \cos nx$, so the derivatives don't converge.

The counter example was easy. What is the correct theorem? Did anyone look it up?

Here is one theorem from Dangello and Seyfried "Introductory Real Analysis" page 177:

<u>Theorem 8.4</u>. Suppose f_n is a sequence of differentiable functions defined on a bounded interval I and that $f_n(x_0)$ converges for some point $x_0 \in I$. If f_n' converges uniformly on I, then f_n converges uniformly on I to a differentiable function F and $F'(x) = \lim_{n\to\infty} f_n'(x)$ for all $x \in I$.

This theorem is also proved in William Wade, "Introduction to Analysis 3rd Ed" as Theorem 7.12 on page 189.

This theorem assumes that f_n' already converge uniformly so it should be lots easier to prove, especially since the claim we tried to prove yesterday was false.

Only 4 people came. On Monday we will discuss whether to continue meeting on Saturday and whether to change the time.

<u>Hints</u> on how to show a monotone function is Borel measurable

We will prove on Monday that if

$$\mathcal{F} = \{ f : \mathbb{R} \to \mathbb{R} \text{ st } f^{-1}((-\infty, a)) \in \mathcal{B} \text{ for } a \in \mathbb{R} \}$$

then $\mathcal{F} = \hat{C}$. Recall that

$$\mathcal{B} = \{ A : \chi_A \in \hat{C} \}$$

and that $\tau \subseteq \mathcal{B}$ where

$$\tau = \{ O \subseteq \mathbb{R} : O \text{ is an open set} \}.$$

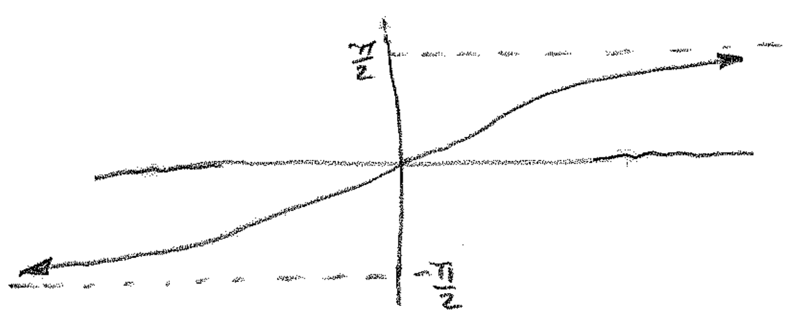
Let's get some intuition by looking at some examples of monotone functions.

Example 1: $f(x) = 1$. Then

$$f^{-1}((-\infty, a)) = \begin{cases} \mathbb{R} & \text{if } a \geq 1 \\ \emptyset & \text{if } a < 1 \end{cases}$$

Since $\mathbb{R}$ and $\emptyset$ are open then $\mathbb{R} \in \mathcal{B}$ and $\emptyset \in \mathcal{B}$. It follows that f is Borel measurable; because this implies $f \in \mathcal{F}$ and from what we will show on Monday, $\mathcal{F} = \hat{C}$.

Example 2: $f(x) = \arctan x$



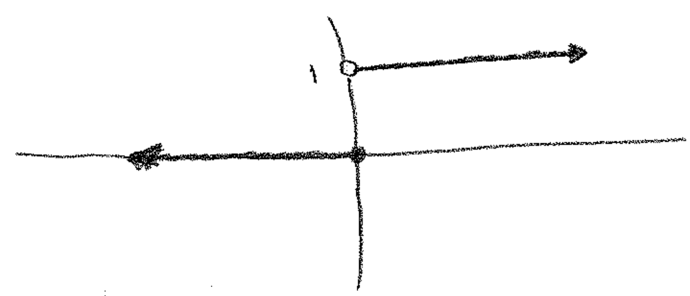
Then

$$f^{-1}((-\infty, a)) = \begin{cases} \mathbb{R} & \text{if } a \geq \pi/2 \\ (-\infty, \tan a) & \text{if } -\frac{\pi}{2} < a < \frac{\pi}{2} \\ \emptyset & \text{if } a \leq -\pi/2 \end{cases}$$

Therefore $f^{-1}((-\infty,a))$ is either $\mathbb{R}$, $(-\infty, \tan a)$ or $\emptyset$. Since $\mathcal{B}$ contains the open sets and all there sets are open then $f^{-1}((-\infty,a)) \in \mathcal{B}$ for any $a \in \mathbb{R}$. It follows that $f \in \mathcal{F} = \hat{C}$.

Example 3

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

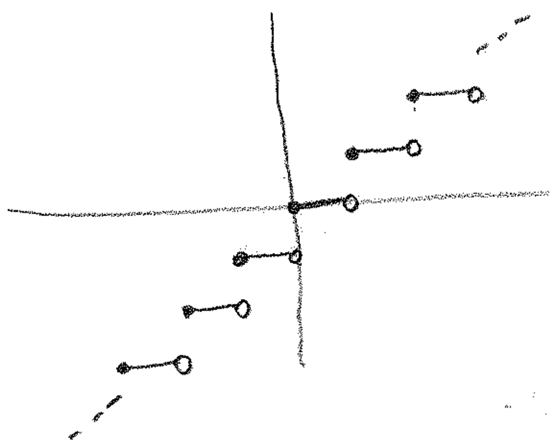


Then

$$f^{-1}((-\infty,a)) = \begin{cases} \mathbb{R} & \text{if } a \geq 1 \\ (-\infty, 1] & \text{if } 0 \leq a < 1 \\ \emptyset & \text{if } a < 0 \end{cases}$$

Since $\mathcal{B}$ is a σ-algebra then it is closed under complements. Thus $\mathcal{B}$ also contains the closed sets. Since $f^{-1}((-\infty,a)) \in \mathcal{B}$ for all $a \in \mathbb{R}$ then $f \in \hat{C}$.

<u>Example 4</u> $f(x) = [\![x]\!]$ the greatest integer less than or equal x. Then



and

$$f^{-1}((-\infty, a)) = (-\infty, -[\![-a]\!])$$

check this

$$f^{-1}((-\infty, 1)) = (-\infty, 1)$$

$$f^{-1}((-\infty, 1.5)) = (-\infty, -[\![-1.5]\!]) = (-\infty, 2)$$

$$f^{-1}((-\infty, -.5)) = (-\infty, -[\![0.5]\!]) = (-\infty, 0)$$

Since $(-\infty, -[\![-a]\!]) \in \mathcal{B}$ for every $a \in \mathbb{R}$ then we have $f \in \hat{C}$.

What do you notice about $f^{-1}((-\infty, a))$ for all the examples discussed?

That $f^{-1}((-\infty, a))$ is some sort of open or closed unbounded interval or $\emptyset$.

If we can prove that $f^{-1}((-\infty, a))$ is always a set of the form

$$\mathbb{R},\ (-\infty, \beta),\ (-\infty, \beta],\ (\alpha, \infty),\ [\alpha, \infty) \text{ or } \emptyset$$

then we have shown that $f \in \mathcal{F}$ and since $\mathcal{F} = \hat{C}$ this implies f is Borel measurable.

All the examples were monotone non-decreasing. If f is monotone non-increasing then $-f$ is monotone non-decreasing. Moreover if $-f \in \hat{C}$, then since $\hat{C}$ is an algebra of funtions it follows that $-(-f) = f \in \hat{C}$. Thus it is sufficient to consider only monotone non decreasing functions.

Claim: If f is monotone non-decreasing then for each $a \in \mathbb{R}$ we have $f^{-1}((-\infty, a))$ is either $\mathbb{R}$, $\emptyset$ or there is some β such that $f^{-1}((-\infty,a))$ is $(-\infty,\beta)$ or $(-\infty,\beta]$.

If this claim is proven this shows that any monotone non-decreasing function is Borel measureable. and consequently that any monotone function is Borel measurable.

Sketch of proof. Given $a \in \mathbb{R}$

If $f^{-1}((-\infty,a)) = \emptyset$ we are done, otherwise let $\beta = \sup f^{-1}((-\infty,a))$

If $\beta = \infty$ claim that $f^{-1}((-\infty,a)) = \mathbb{R}$

... details ...

If $\beta < \infty$ claim that

$$(-\infty,\beta) \subseteq f^{-1}((-\infty,a)) \subseteq (-\infty,\beta]$$

... details ...

and therefore $f^{-1}((-\infty,a))$ is either $(-\infty,\beta)$ or $(-\infty,\beta]$.

An alternative approach would be to show that a monotone function is Borel measurable directly using the definition of $\hat{C}$.

<u>Recall</u>: $\hat{C}$ is the smallest collection of functions closed under pointwise limits that contains the continuous functions.

In particular if f_n are continuous and $f_n \to f$ pointwise then $f \in \hat{C}$. Note, however, there may be functions in $\hat{C}$ that are not the pointwise limit of continuous functions. Inductively we could define

$$S_0 = C$$

$$S_{k+1} = \left\{ f : \mathbb{R} \to \mathbb{R} \text{ s.t. } \begin{array}{l} f_n \to f \text{ pointwise for} \\ \text{some sequence } f_n \in S_k \end{array} \right\}$$

Then $S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots$ and $S_0 \neq S_1 \neq S_2 \neq \cdots$.

<u>Question</u>: Is $\hat{C} = \bigcup_{k=0}^{\infty} S_k$?

It is possible to construct $\hat{C}$ using transfinite induction. This may help answer the above question.

Another question: Let

$$\mathcal{M} = \{ f : \mathbb{R} \to \mathbb{R} \text{ st } f \text{ is monotone} \}.$$

We know that $\mathcal{M} \subseteq \hat{C}$.

Is it true that $\mathcal{M} \subseteq S_1$?

If not, is $\mathcal{M} \subseteq S_2$?

This is a more difficult question than the homework question. Still it is interesting. I almost wish I had put is as extra credit on the next homework. Now if $\mathcal{M} \subseteq S_1$ this would imply $\mathcal{M} \subseteq \hat{C}$, but how could one show $\mathcal{M} \subseteq S_1$.

Suppose $$f(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

then $$f_n(x) = \begin{cases} 1 & \text{for } x > \frac{1}{n} \\ nx & \text{for } 0 \leq x \leq \frac{1}{n} \\ 0 & \text{for } x \leq 0 \end{cases}$$

are continuous and $f_n \to f$ pointwise

On the other hand if

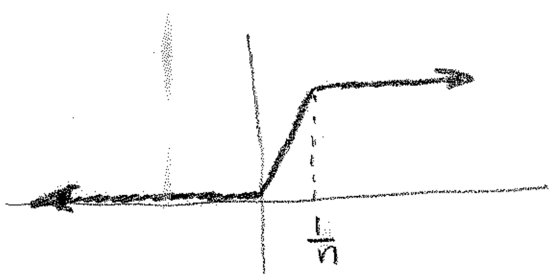
$$f(x) = \begin{cases} 1 & \text{for} \quad x \geq 0 \\ 0 & \text{for} \quad x < 0 \end{cases}$$

then

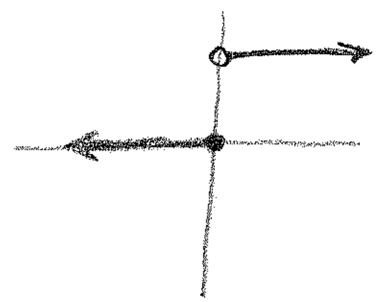
$$f_n(x) = \begin{cases} 1 & \text{for} \quad x \geq 0 \\ 1 - nx & \text{for} \quad -\frac{1}{n} < x < 0 \\ 0 & \text{for} \quad x \leq \frac{1}{n} \end{cases}$$

are continuous and $f_n \to f$ pointwise.

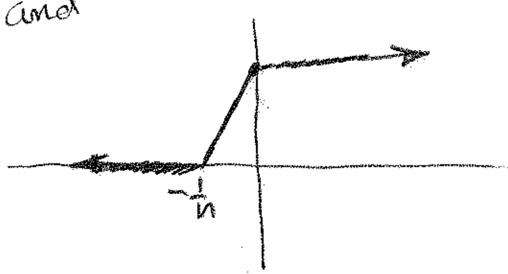
In pictures these are two cases.



converges to



and



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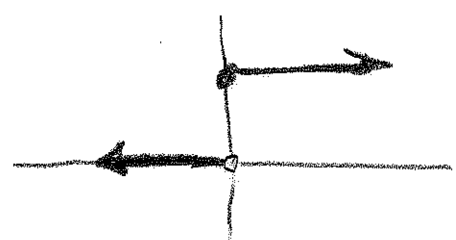
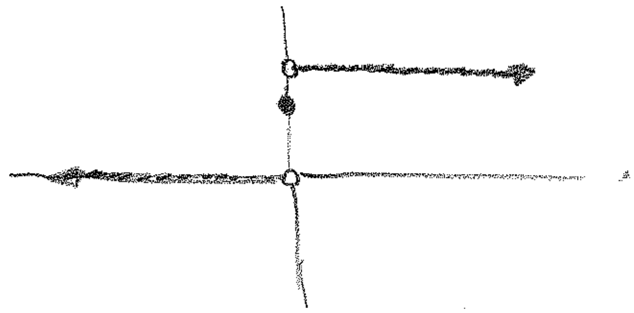


There are other kinds of jump discontinuities a monotone function might have.

For example



We know from a similar proof to the extra credit problem on the previous homework that the set of jump discontinuities is countable.

Is it possible to create a sequence of functions using some sort of diagonalization or induction argument to show any monotone function is the pointwise limit of continuous functions. On the other hand, just treating one jump discontinuity involved a number of different cases so perhaps there are monotone functions which are not the pointwise limit of continuous functions.

It might be interesting to try searching on the internet or posting in one of the mathematics and science forums to find an answer.

We know $\mathbb{R}$ is uncountable and that the polynomials $\mathbb{P} = \bigcup_{n=1}^{\infty} \mathbb{P}_n$ where $\mathbb{P}_n$ is the set of polynomials with degree less than or equal n.

Since each $\mathbb{P}_n \sim \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n+1 \text{ times}} \sim \mathbb{R}$

then is it true that

$$\mathbb{P} = \bigcup_{n=1}^{\infty} \mathbb{P}_n \sim \mathbb{R}$$

Since each function in C may be written as the limit of polynomials is it true that $C \sim \mathbb{R}$?

If so, what about $\hat{C}$?

Is $\hat{C} \sim \mathbb{R}$?

These questions are related to the study of ordinal and cardinal numbers. Some analysis books spend more time than ours on this topic. You may also study ordinal and cardinal number is a set theory or logic course