Math 713 Summary

Oct 18, '10

Recall

$\tau = \{ O \subseteq \mathbb{R} : O \text{ is open} \}$

$C = \{ f : \mathbb{R} \to \mathbb{R} \text{ s.t. } f^{-1}(O) \in \tau \text{ for every } \tau \in O \}$

We have defined

$\hat{C}$ = the smallest collection of functions ~~that~~ contains the continuous functions.

$\mathcal{B} = \{ B \subseteq \mathbb{R} : \chi_B \in \hat{C} \}$.

We are trying to show that

$\mathcal{B}$ = the smallest σ-algebra that contains τ.

$\hat{C} = \{ f : \mathbb{R} \to \mathbb{R} \text{ st } f^{-1}(A) \in \mathcal{B} \text{ for every } A \in \mathcal{B} \}$.

To this end we have defined

$\mathcal{F}_1 = \{ f : \mathbb{R} \to \mathbb{R} \text{ st } f^{-1}(B) \in \mathcal{B} \text{ for every } B \in \mathcal{B} \}$

$\mathcal{F}_2 = \{ f : \mathbb{R} \to \mathbb{R} \text{ st } f^{-1}(O) \in \mathcal{B} \text{ for every } O \in \tau \}$

$\mathcal{F}_3 = \{ f : \mathbb{R} \to \mathbb{R} \text{ st } f^{-1}((-\infty, a)) \in \mathcal{B} \text{ for every } a \in \mathbb{R} \}$

$\mathcal{F}_4 = \{ f : \mathbb{R} \to \mathbb{R} \text{ s.t. } f^{-1}((a, \infty)) \in \mathcal{B} \text{ for every } a \in \mathbb{R} \}$

So far we have shown that $\mathcal{F}_2 = \mathcal{F}_3 = \mathcal{F}_4$. The book calls this set $\mathcal{F}$.

We know that $\mathcal{F}$ contains the continuous functions, what is next is to show $\mathcal{F}$ is closed under pointwise limits. First, lets recall the lemma proved last time

<u>Lemma</u>: Let $g_n : \mathbb{R} \to \mathbb{R}$ be a sequence of functions and define $g = \sup\{g_n : n \in \mathbb{N}\}$. Then

$$g^{-1}((a,\infty)) = \bigcup_{n=1}^{\infty} g_n^{-1}((a,\infty)).$$

Now given $G_n : \mathbb{R} \to \mathbb{R}$ and define $G = \inf\{G_n : n \in \mathbb{N}\}$. Set $g_n = -G_n$. Then

$$g = \sup\{g_n : n \in \mathbb{N}\} = \sup\{-G_n : n \in \mathbb{N}\}$$
$$= -\inf\{G_n : n \in \mathbb{N}\} = -G$$

It follows that

$$G^{-1}((-\infty,a)) = \{x : G(x) < a\} = \{x : -g(x) < a\}$$
$$= \{x : g(x) > -a\} = g^{-1}((-a,\infty))$$
$$= \bigcup_{n=1}^{\infty} g_n^{-1}((-a,\infty)) = \bigcup_{n=1}^{\infty} \{x : g_n(x) > -a\}$$
$$= \bigcup_{n=1}^{\infty} \{x : -G_n(x) > -a\} = \bigcup_{n=1}^{\infty} \{x : G_n(x) < a\}$$
$$= \bigcup_{n=1}^{\infty} G_n^{-1}((-\infty,a)).$$

Therefore we have proved that

<u>Lemma</u> Let $G_n : \mathbb{R} \to \mathbb{R}$ be a sequence of functions and define $G = \inf\{G_n : n \in \mathbb{N}\}$. Then

$$G^{-1}((-\infty,a)) = \bigcup_{n=1}^{\infty} G_n^{-1}((-\infty,a)).$$

Comparing these two lemmas, one might wonder if

$$g^{-1}((-\infty,a)) = \bigcap_{n=1}^{\infty} g_n^{-1}((-\infty,a))$$

and

$$G^{-1}((a,\infty)) = \bigcap_{n=1}^{\infty} G_n^{-1}((a,\infty)).$$

So are there true or false?

True	False
7	8

This is not decisive. Did anyone who voted false have a counterexample?

No?

Did anyone who voted true have a proof?

YES?

Really?

The proposed proof is

$$g^{-1}((-\infty,a)) = g^{-1}([a,\infty))^c = \left(\bigcup_{n=1}^{\infty} g_n^{-1}([a,\infty))\right)^c$$

$$= \bigcap_{n=1}^{\infty} g_n^{-1}([a,\infty))^c = \bigcap_{n=1}^{\infty} g_n^{-1}((-\infty,a)).$$

Does everyone believe this proof?

Does anyone see a possible problem?

In the second equality it is claimed

$$g^{-1}([a,\infty)) = \bigcup_{n=1}^{\infty} g_n^{-1}([a,\infty))$$

This looks like last weeks lemma

$$g^{-1}((a,\infty)) = \bigcup_{n=1}^{\infty} g_n^{-1}((a,\infty))$$

except with a closed interval $[a,\infty)$ rather that (a,∞).

Is the lemma still true for closed intervals?

Look at the proof from last week. Did it use the fact that the interval is open?

~~If so is there a way to get around that assumption?~~

Please think about this and we will talk about it next week.

Proposition. If $g_n \in \mathcal{F}$ and $g = \sup\{g_n : n \in \mathbb{N}\}$, then $g \in \mathcal{F}$. Similarly, If $G_n \in \mathcal{F}$ and $G = \inf\{G_n : n \in \mathbb{N}\}$, then $G \in \mathcal{F}$.

Proof: Since $g_n \in \mathcal{F} = \mathcal{F}_4$ then $g_n^{-1}((a,\infty)) \in \mathcal{B}$. Now since $\mathcal{B}$ is a σ-algebra then $\bigcup_{n=1}^{\infty} g_n^{-1}((a,\infty)) \in \mathcal{B}$. By the previous lemma $\bigcup_{n=1}^{\infty} g_n^{-1}((a,\infty)) = g^{-1}((a,\infty))$. Therefore $g^{-1}((a,\infty)) \in \mathcal{B}$. The implies $g \in \mathcal{F}_4 = \mathcal{F}$.

The proof that $G \in \mathcal{F}$ is similar.

Proposition: If $f_n \in \mathcal{F}$ and $f_n \to f$ pointwise, then $f \in \mathcal{F}$.

Proof: Define $G_n = \sup\{f_k : k \geq n\}$. By the above proposition $G_n \in \mathcal{F}$. Now let $G = \inf\{G_n : n \in \mathbb{N}\}$. Since $G_n \in \mathcal{F}$ the second part of the above proposition shows $G \in \mathcal{F}$. Now

$$f(x) = \lim_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x)$$

$$= \lim_{n\to\infty} \sup\{f_k(x) : k \geq n\} = \inf\{\sup\{f_k(x) : k \geq n\} : n \in \mathbb{N}\}$$

$$= \inf\{G_n(x) : n \in \mathbb{N}\} = G(x)$$

implies that $f \in \mathcal{F}$.

Therefore $\mathcal{F}$ is a set which contains the continuous functions and is closed under pointwise limits.

Since $\hat{C}$ is the smallest such set, it follows that

$$\hat{C} \subseteq \mathcal{F}.$$

In order to show $\hat{C} = \mathcal{F}$ we need to prove

Lemma $\mathcal{F} \subseteq \hat{C}$.

Proof: Let $f \in \mathcal{F}$. For $k \in \mathbb{Z}$ and $n \in \mathbb{N}$

define $E_{n,k} = f^{-1}\left(\left[\frac{k}{n}, \frac{k+1}{n}\right)\right)$

Claim $E_{n,k} \in \mathcal{B}$

At this point the author of or book writes "why" in parenthesis. Although it sounds like the author doesn't know why, this is actually a euphemism which means the author has left out some details which he expects you can figure out on your own.

Lemma: $[a,b) \in \mathcal{B}$.

Let $x \in (a,b)$. Since $[a,x]$ is closed then $[a,x]^c$ is open. We know that $\mathcal{B}$ contains the open sets therefore $[a,x]^c \in \mathcal{B}$. Since $\mathcal{B}$ is a σ-algebra this implies $([a,x]^c)^c = [a,x] \in \mathcal{B}$.

Also (x,b) is open so $(x,b) \in \mathcal{B}$.

Since $\mathcal{B}$ is a σ-algebra it follows that

$$[a,x] \cup (x,b) = [a,b) \in \mathcal{B}.$$

Can you modify the proof of this lemma to obtain

Lemma: If $f \in \mathcal{L}$ then $f^{-1}([a,b)) \in \mathcal{B}$

This would answer the "why" in the proof of the proposition we are working on. Since ~~there is no time to say more about this today~~ we will finish the proof assuming this lemma.

Recall, we a proving $\mathcal{F} \subseteq \hat{C}$.

Let $f \in \mathcal{F}$ and $E_{n,k} = f^{-1}([\frac{k}{n}, \frac{k+1}{n}))$.

By the lemma $E_{n,k} \in \mathcal{B}$.

Therefore $\chi_{E_{n,k}} \in \hat{C}$.

Since $\hat{C}$ is an algebra then

$$f_{n,m} = \sum_{k=-m}^{m} \frac{k}{n} \chi_{E_{n,k}} \in \hat{C}.$$

Since $\hat{C}$ is closed under pointwise limits then

$$\lim_{m \to \infty} f_{n,m}(x) = \sum_{k=-\infty}^{\infty} \frac{k}{n} \chi_{E_{n,k}}(x)$$

shows that $f_n = \sum_{k=-\infty}^{\infty} \frac{k}{n} \chi_{E_{n,k}} \in \hat{C}$.

Claim $f_n \to f$ as $n \to \infty$. The book says in parenthesis that, in fact, $f_n \to f$ uniformly.

After we have justified this claim then since $f_n \in \hat{C}$ and $\hat{C}$ is closed under pointwise and therefore uniform limits it follows that $f \in \hat{C}$.

Proof of the claim $f_n \to f$ uniformly.

Given $\varepsilon > 0$ choose $N \in \mathbb{N}$ so large $\frac{1}{N} < \varepsilon$. Then for $n \geq N$ and $x \in \mathbb{R}$ there is $k_0 \in \mathbb{Z}$ such that $f(x) \in \left[\frac{k_0}{n}, \frac{k_0+1}{n}\right)$.

This implies $x \in f^{-1}\left(\left[\frac{k_0}{n}, \frac{k_0+1}{n}\right)\right) = E_{n,k_0}$.

Since $f_n(x) = \sum_{k=-\infty}^{\infty} \frac{k}{n} \chi_{E_{n,k}}(x) = \frac{k_0}{n}$

We obtain that

$$|f(x) - f_n(x)| = f(x) - f_n(x) = f(x) - \frac{k_0}{n}$$

$$< \frac{k_0+1}{n} - \frac{k_0}{n} = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

which justifies the claim.

Thus, except for proving the lemma which states that $f^{-1}([a,b)) \in \mathcal{B}$, we have proven that $\mathcal{F} \subseteq \hat{C}$. It follows that $\mathcal{F} = \hat{C}$.

Please look at the construction used in this proof to obtain the functions f_n which are uniformly convergent to f. This is an important construction which we may use in the future.

We are now ready to show that

$$\mathcal{B} \equiv \text{the smallest } \sigma\text{-algebra that contains } \tau$$

Let $\mathcal{A}$ be any σ-algebra that contains τ.

<u>Claim</u> $\mathcal{B} \subseteq \mathcal{A}$.

Define $\mathcal{Y} = \{ f: \mathbb{R} \to \mathbb{R} \text{ s.t. } f^{-1}(O) \in \mathcal{A} \text{ for every } O \in \tau \}$.

Since $\tau \subseteq \mathcal{A}$ then $C \subseteq \mathcal{Y}$. Moreover $\mathcal{Y}$ is closed under pointwise limits for the same reason that $\mathcal{F}$ is closed under pointwise limits. Since $\hat{C}$ is the smallest collection of functions containing C which is closed under pointwise limits, then $\hat{C} \subseteq \mathcal{Y}$.

Now let $B \in \mathcal{B}$. Then $\chi_B \in \hat{C}$. Since $\hat{C} \subseteq \mathcal{Y}$, this means $\chi_B \in \mathcal{Y}$. Therefore taking $O = (\frac{1}{2}, \frac{3}{2})$

$$\chi_B^{-1}((\tfrac{1}{2}, \tfrac{3}{2})) = B \in \mathcal{A}.$$

This shows that $\mathcal{B} \subseteq \mathcal{A}$ which implies $\mathcal{B}$ is the smallest σ-algebra that contains τ.