Math 713 Summary $\qquad$ Oct 20, 2010

Let $\mathcal{F}_1 = \{f : \mathbb{R} \to \mathbb{R}$ st $f^{-1}(B) \in \mathcal{B}$ for every $B \in \mathcal{B}\}$

$\mathcal{F}_2 = \{f : \mathbb{R} \to \mathbb{R}$ st $f^{-1}(O) \in \mathcal{B}$ for every $O \in \tau\}$

$\mathcal{F}_3 = \{f : \mathbb{R} \to \mathbb{R}$ st $f^{-1}((-\infty, a)) \in \mathcal{B}$ for every $a \in \mathbb{R}\}$

$\mathcal{F}_4 = \{f : \mathbb{R} \to \mathbb{R}$ st $f^{-1}((a, \infty)) \in \mathcal{B}$ for every $a \in \mathbb{R}\}$

We have shown that

$$\mathcal{F}_2 = \mathcal{F}_3 = \mathcal{F}_4 = \hat{C}$$

and $\mathcal{B} = \mathcal{A}(\tau)$

<u>Theorem</u>: $\mathcal{F}_1 = \hat{C}$.

Since the conditions for f to be in $\mathcal{F}_1$ are more restrictive than for f to be in $\mathcal{F}_2$ then $\mathcal{F}_1 \subseteq \mathcal{F}_2$.

It remains to show $\mathcal{F}_2 \subseteq \mathcal{F}_1$. Let $f \in \mathcal{F}_2$ and define $\mathcal{A} = \{A \subseteq \mathbb{R} : f^{-1}(A) \in \mathcal{B}\}$.

Since $\mathcal{B}$ is a σ-algebra it follows from one of the lemmas of last week that $\mathcal{A}$ is a σ-algebra. Specifically, look at page 3 of the lecture notes from October 14.

Since $f \in \mathcal{F}_2$ then $f^{-1}(O) \in \mathcal{B}$ for every $O \in \tau$.

It follows that $\tau \subseteq \mathcal{A}$.

Therefore $\mathcal{A}$ is a σ-algebra that contains τ.

We showed last time that $\mathcal{B}$ was the smallest σ-algebra that contains τ.

Thus $\mathcal{B} \subseteq \mathcal{A}$.

This means that $f^{-1}(B) \in \mathcal{B}$ for every $B \in \mathcal{B}$, or, in otherwords, that $f \in \mathcal{F}_1$.

The only thing left from section 3.1 is to go back and prove the lemma

<u>lemma</u>: If $f \in \mathcal{F}$ then $f^{-1}([a,b)) \in \mathcal{B}$.

Since we already know $[a,b) \in \mathcal{B}$ then can we just use the above theorem to conclude that $f \in \mathcal{F} = \mathcal{F}_1$ implies $f^{-1}([a,b)) \in \mathcal{B}$?

Or is this argument circular?

It might be circular because we used

$$\mathcal{B} = \mathcal{A}(\tau)$$

to prove $\mathcal{F}_1 = \mathcal{F}$,

and we used

$$\mathcal{F} = \hat{C}$$

to prove $\mathcal{B} = \mathcal{A}(\tau)$,

and we used

$$f^{-1}([a,b)) \in \mathcal{B}$$

to prove $\mathcal{F} = \hat{C}$.

Therefore we need to prove $f^{-1}([a,b))$ without using these results. In fact the proof is easy and only a simple change to the proof of the fact that $[a,b) \in \mathcal{B}$.

Let $x \in (a,b)$. Since $f \in \mathcal{F}_2$ then $f^{-1}([a,x]^c) \in \mathcal{B}$. Since $\mathcal{B}$ is a σ-algebra $f^{-1}([a,x]^c)^c = f^{-1}([a,x]) \in \mathcal{B}$. Also $f^{-1}((x,b)) \in \mathcal{B}$. It follows that

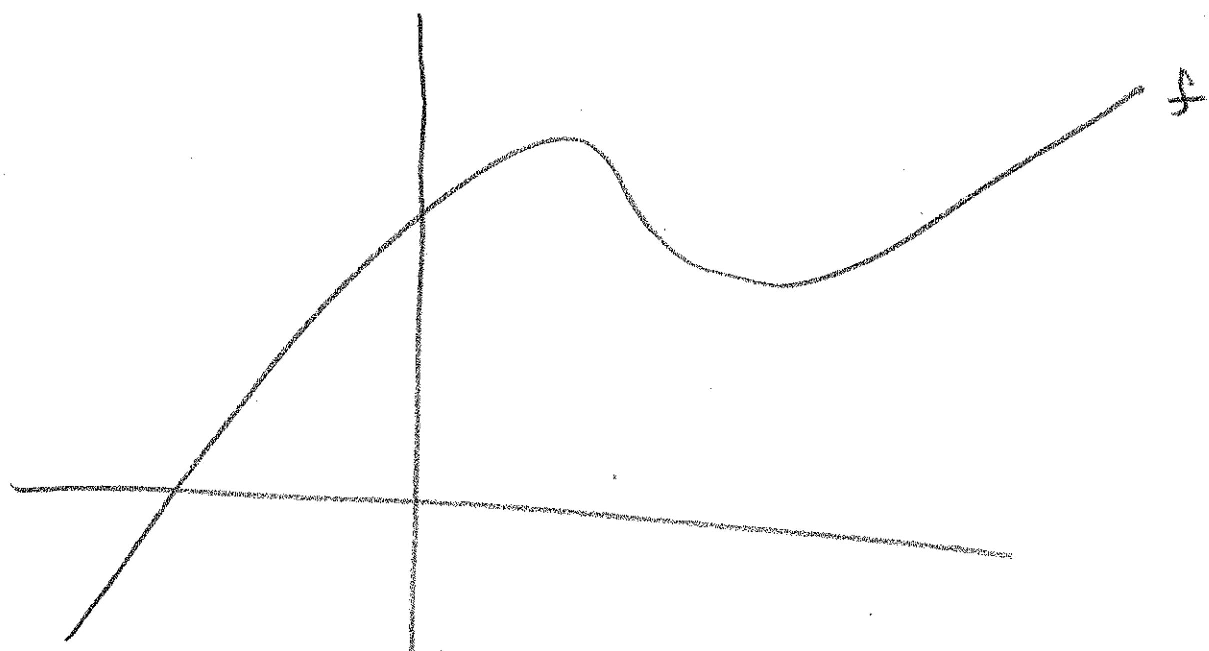
$$f^{-1}([a,x]) \cup f^{-1}((x,b)) = f^{-1}([a,b)) \in \mathcal{B}.$$

As stated last time the construction used to create the uniformly convergent sequence f_n is important. Let work through some examples of this construction.

Let $E_{n,k} = f^{-1}\left(\left[\frac{k}{n}, \frac{k+1}{n}\right)\right)$

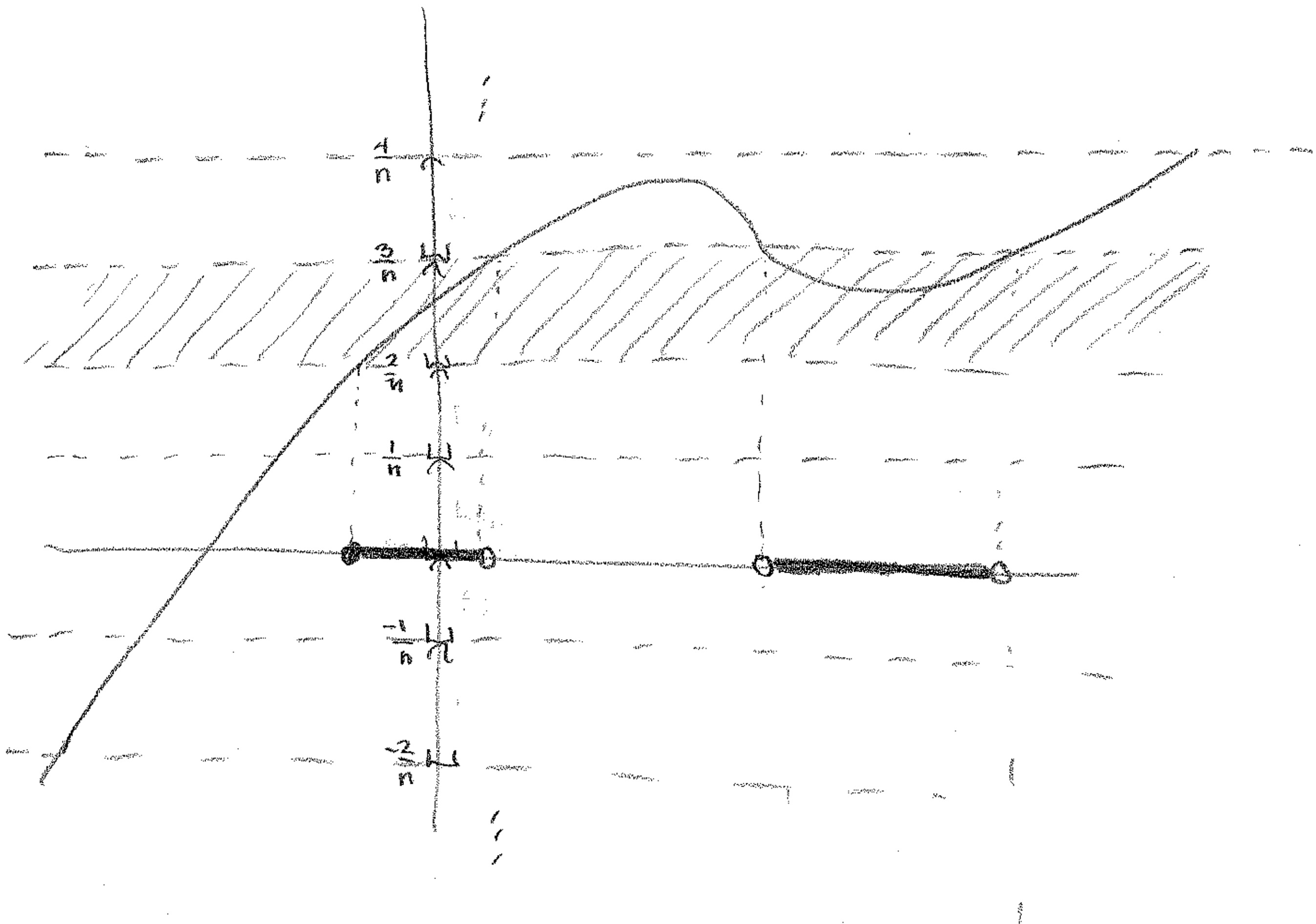
$$f_n = \sum_{k=-\infty}^{\infty} \frac{k}{n} \chi_{E_{n,k}}.$$

The function f is given by the following graph. What is f_n?

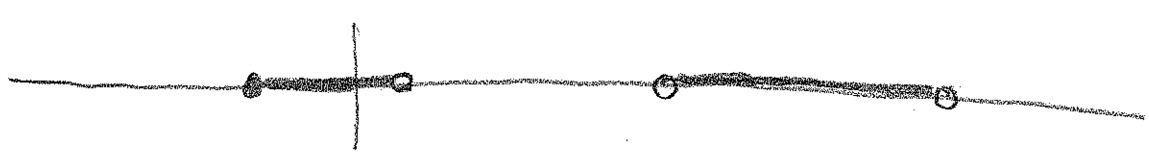


We know f_n is less than f.

Divide the y-axis into the sets $[\frac{k}{n}, \frac{k+1}{n})$

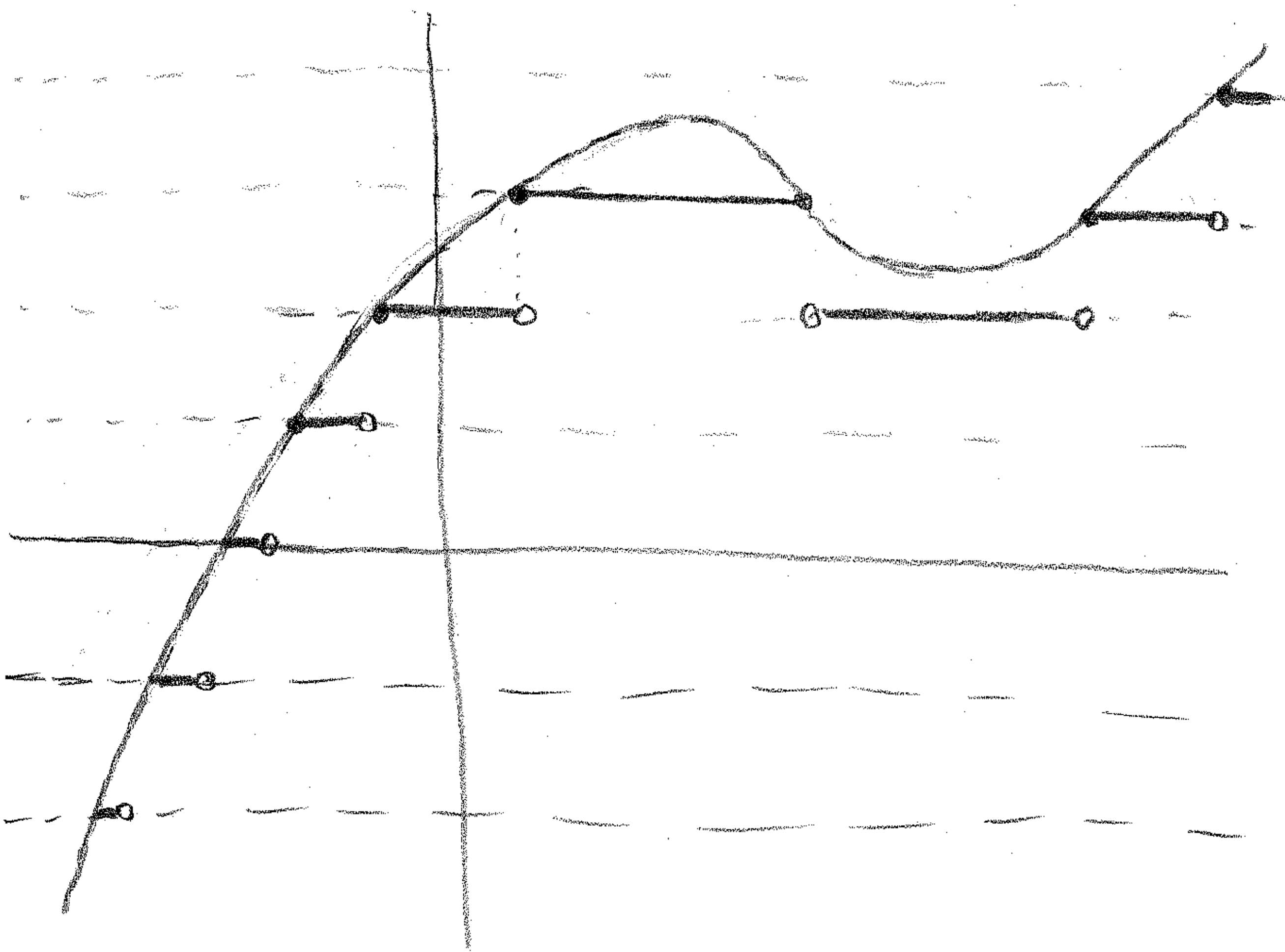


Choose some value of k and find $E_{n,k}$. For example $k=2$. Then we shade the set $[\frac{2}{n}, \frac{3}{n})$ above and to inverse image to find $E_{n,2}$. Thus $E_{n,2}$ is the set



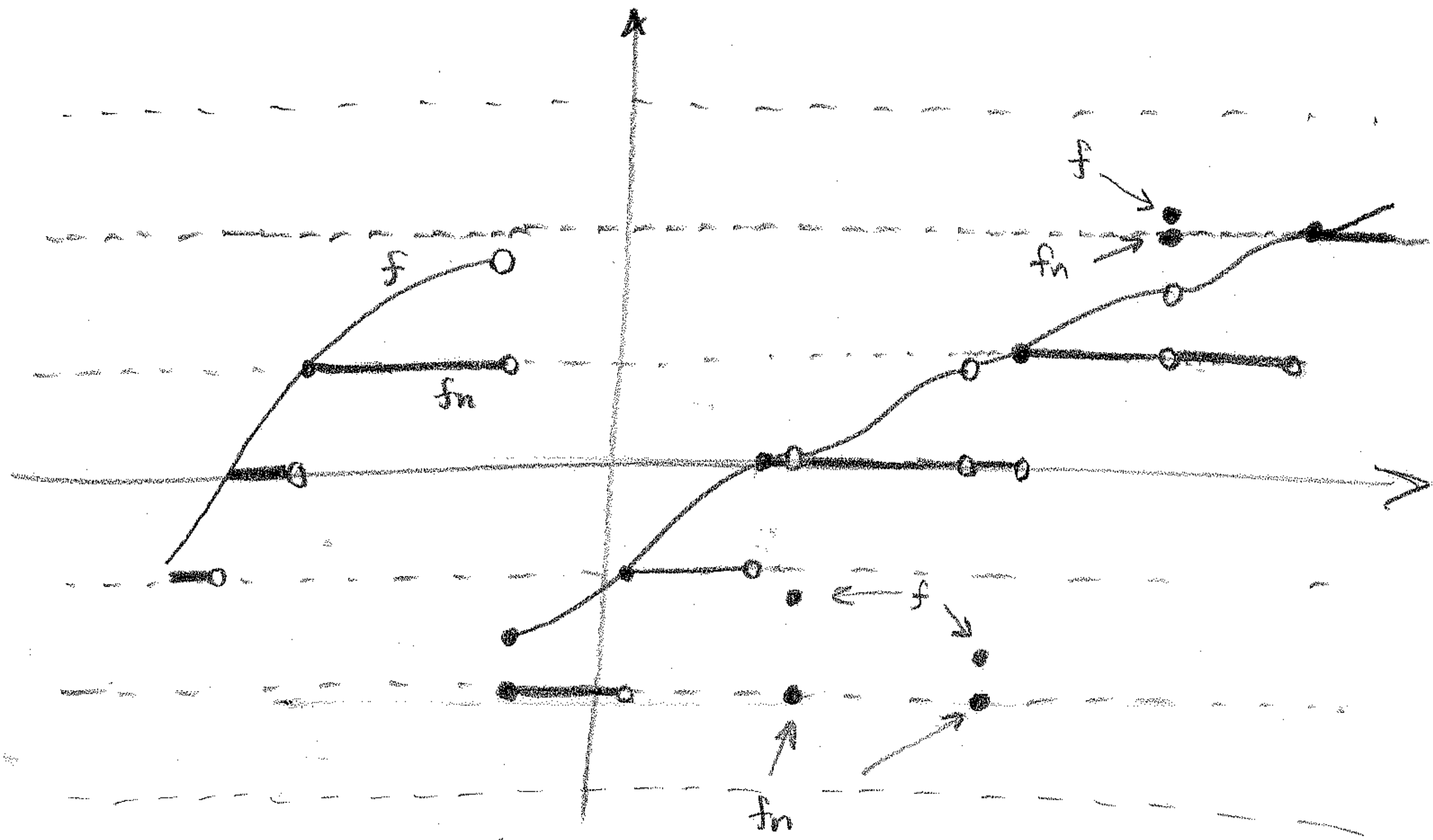
of x values.

The function $\frac{k}{n}\chi_{E_{n,k}}$ take the set of x-values in $E_{n,k}$ and gives them y-values with height $\frac{k}{n}$. Doing this for each value of k and adding them up gives f_n.



note that f_n was colored in pink chalk on the black board but due to limitations in technology presented here in black.

Let's work another example, this time with discontinuities.



If you took a photograph of the graph with the different colors, send it to me and I will post it with these lecture notes.

It should be observed that by dividing the graph into horizontal strips that every point on the approximation is no more that a distance $\frac{1}{n}$ from the corresponding point of f.

In advanced calculus or undergraduate analysis you may recall the theorem

Theorem: If $f_n \to f$ uniformly on $[a,b]$ and each f_n is Riemann integrable then f is Riemann integrable and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n .$$

This result is proved, for example, as Theorem 8.3 in Dangello and Seyfried, see also Theorem 7.10 in Wade.

Thus the kind of approximations that we have constructed by

$$f_n = \sum_{k=-\infty}^{\infty} \frac{k}{n} \chi_{E_{n,k}}$$

can be expected to behave well with respect to integration because the converge uniformly.

This notion of approximating the function uniformly by dividing the graph of the function into horizontal strips can be contrasted with the upper and lower approximations used by Riemann.

The step functions used in the Riemann theory of integration divide the graph into vertical strips.

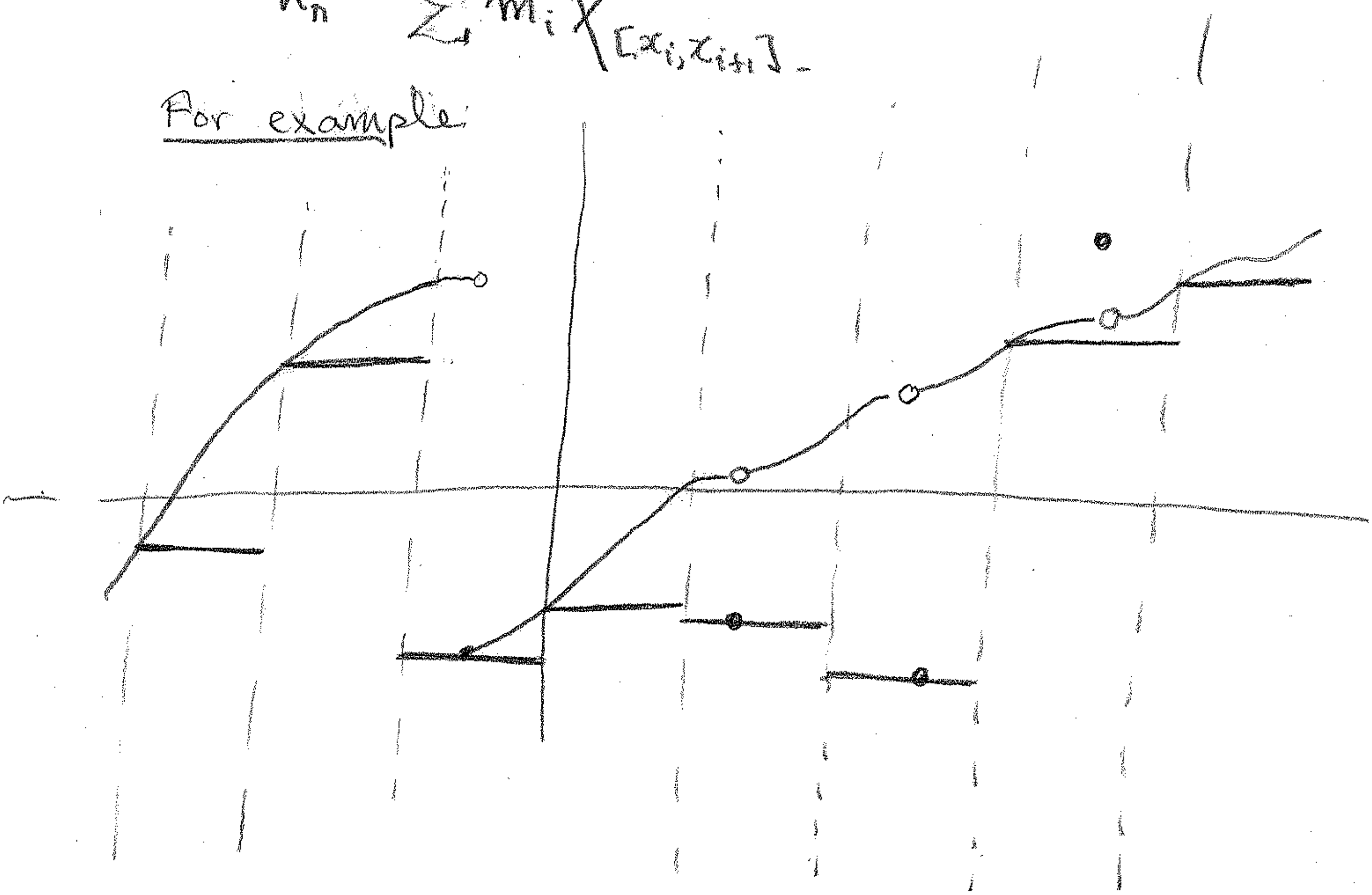
The lower sum is given by

$$\sum m_i \Delta x_i \quad \text{where} \quad m_i = \inf\{f(x) : x \in [x_i, x_{i+1}]\}.$$

This is what is obtained by the approximation of f by the step function.

$$h_n = \sum m_i \chi_{[x_i, x_{i+1}]}.$$

For example:



It is clear that this approximation is far from uniform. In fact h_n may not even converge pointwise to f as $n \to \infty$.

In the Riemann theory, the points of discontinuity have to constitute a small set so they can be treated in a special way when showing the upper and lower Riemann integrals are equal.

The Lebesgue theory of integration avoids this requirement by approximating the functions in a way that the points of discontinuity do not affect the convergence of the approximations.

In particular the Lebesgue theory that will be developed in the next sections allows function that are discontinuous at every point to be integrated.