Math 713 Recitation    Oct 22-23, '10

What's going to happen next?

We are developing $\sigma$-algebras and measures so that the approximations

$$f_n = \sum_{k=-\infty}^{\infty} \frac{k}{n} \chi_{E_{nk}}$$

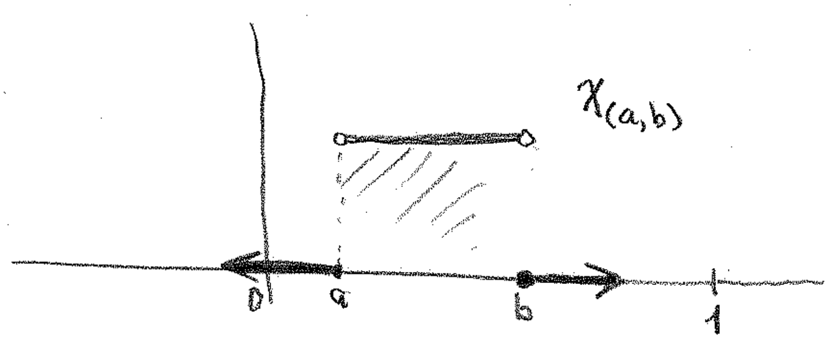
where $E_{n,k} = f^{-1}\left(\left[\frac{k}{n}, \frac{k+1}{n}\right)\right)$ can be integrated.

Since these approximations converge uniformly to $f$ this allows the integral of $f$ to be defined by

$$\int_0^1 f = \lim_{n\to\infty} \int_0^1 f_n$$

How to integrate a characteristic function?

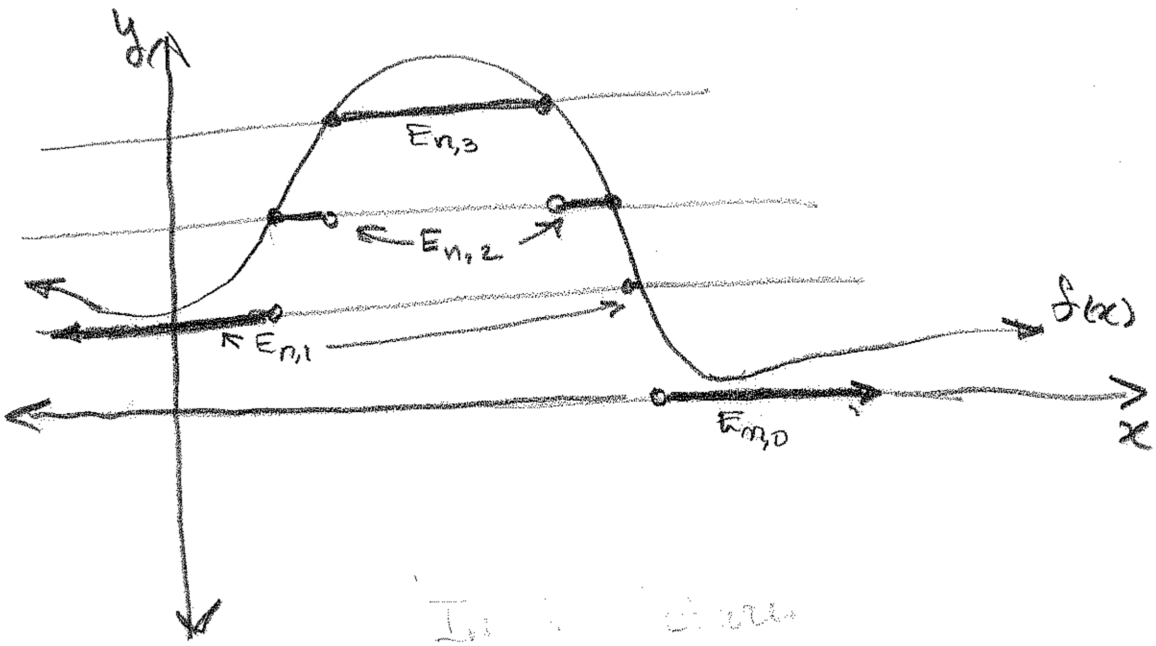
If the set is an interval, then it is easy.



So it is obvious that $(a,b) \subseteq [0,1]$ and

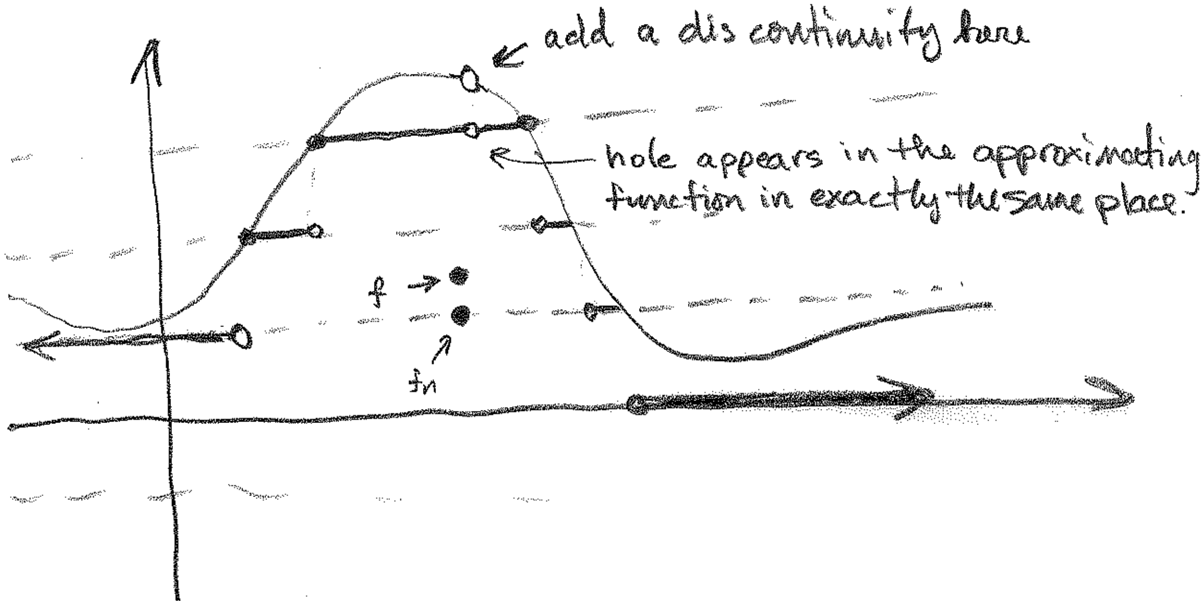
$$\int_0^1 \chi_{(a,b)} = 1 \cdot \ell((a,b)) = b-a.$$

Generally our sets $E_{n,k}$ will not be intervals

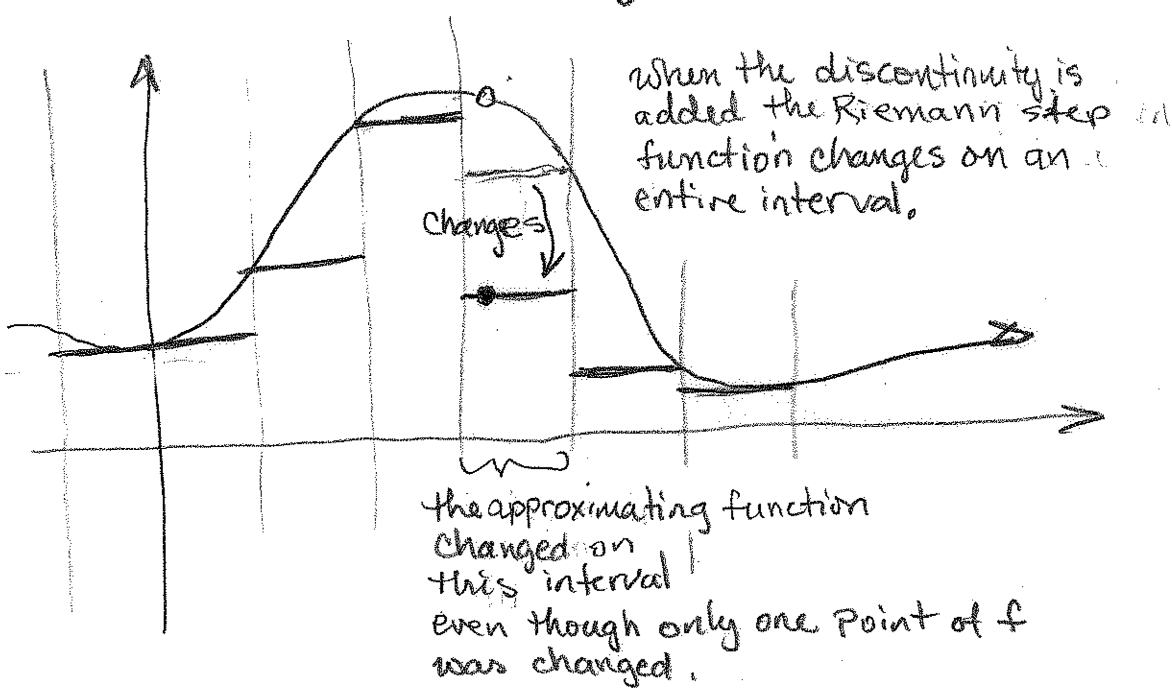



In the picture both $E_{n,0}$ and $E_{n,3}$ are intervals but $E_{n,1}$ and $E_{n,2}$ are unions of two intervals. If $f$ were a function with discontinuities the sets $E_{n,k}$ could be much more complicated. In fact if $f \in \hat{C}$, the most we can say about $E_{n,k}$ are that $E_{n,k} \in \mathcal{B}$.

Note that the approximations $f_n$ are uniform and if you add a discontinuity to $f$ it doesn't affect how well $f_n$ approximates $f$.




The approximating sums used in the Riemann integral behave differently



The advantage of the Lebesgue construction is that it handles discontinuities better. In the Riemann theory the discontinuity result in an approximation of $f$ that is non-uniform. Indeed if you take the partition for the Riemann sum small enough the interval containing the point of discontinuity can be made $\varepsilon$ small, but with the Lebesque construction no $\varepsilon$ needs to be introduced to handle the discontinuity. As a result, it is posible to integrate a function that is everywhere discontinuous using the Lebesque integral.

Of course the Riemann integral must have some advantages as well, or we wouldn't still be teaching it. One advantage of the Riemann integral is that the form of the approximating function allows proving things like the fundamental theorem of Calculus from the mean value theorem. It also avoids the $\sigma$-algebras and measures.

The need to develop an integral that can integrate a wider class of functions is useful in applications. In an application we may obtain a function $f$ as the limit of some functions $f_n$. Since $\hat{C}$ is closed under pointwise limits then we know that not matter what $f$ is that $f \in \hat{C}$.

Sometimes, knowing that $f \in \hat{C}$ exists is enough to do additional analysis that might eventually show $f$ is even continuous and therefore Riemann integrable. In some ways this is like using limsup and liminf to show a limit exists.

Can anyone show that if $f_n : [0,1] \to \mathbb{R}$ is a sequence of Riemann integrable functions such that $f_n \to f$ uniformly then $f$ is Riemann integrable and $\int_0^1 f_n \to \int_0^1 f$ ?

My recollection of the proof is that most of the effort is involved in showing $f$ is, in fact, Riemann integrable.

To make things easier let's try to prove a simpler result first:

If $f_n : [0,1] \to \mathbb{R}$ is a sequence of continuous functions such that $f_n \to f$ uniformly, then $f$ is continuous and $\int_0^1 f_n \to \int_0^1 f$.

The fact that $f$ is continuous was a problem on the first quiz. Therefore, the only thing left is to show that $\int_0^1 f_n \to \int_0^1 f$.

Let $\varepsilon > 0$. Since $f_n \to f$ uniformly, there is $N \in \mathbb{N}$ such that $n \geq N$ implies

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } x \in [0,1].$$

Now

$$\left| \int_0^1 f_n - \int_0^1 f \right| \leq \int_0^1 |f_n - f| < \int_0^1 \varepsilon = \varepsilon$$

shows that $\int_0^1 f_n \to \int_0^1 f$ as $n \to \infty$.

Does anyone want to criticize or elaborate on the above proof?

Remark 1:

Since $f_n \in C$ and $f \in C$ then $f_n$ and $f$ are Riemann integrable so $\int_0^1 f_n$ and $\int_0^1 f$ make sense.

Remark 2:

Can you prove that if $f$ and $g$ are Riemann integrable

and

$f(x) < g(x)$ for all $x \in [0,1]$

then $\int_0^1 f < \int_0^1 g$?

In the case that $f$ and $g$ are continuous then this result is almost obvious, but what about if these functions are discontinuous?

What if $f$ and $g$ are only assumed to be integrable in the Lebesgue sense?

In the proof we assumed $f_n$ was continuous and so $f$ was continuous and therefore

$$|f_n - f|$$

is continuous. Obviously the constant function

$$\varepsilon$$

is continuous, so

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for } x \in [0,1]$$

clearly implies

$$\int_0^1 |f_n(x) - f(x)| < \int_0^1 \varepsilon .$$

However, the assumption of continuity was to simply things and allow us to get started on a problem. The original question assumed that $f_n$ was only Riemann integrable.

An alternative approach would be to modify the proof so the preservation of the strict inequality is not needed.

For example

Let $\varepsilon > 0$. Since $f_n \to f$ uniformly, then there is $N \in \mathbb{N}$ such that $n \geq N$ implies

$$|f_n(x) - f(x)| < \varepsilon/2 \quad \text{for all } x \in [0,1].$$

Then

$$\left| \int_0^1 f_n - \int_0^1 f \right| \leq \int_0^1 |f_n - f| \leq \int_0^1 \varepsilon/2 = \varepsilon/2 < \varepsilon$$

shows that $\int_0^1 f_n \to \int_0^1 f$ as $n \to \infty$.

Let's think now how to prove that if $f$ and $g$ are Riemann integrable such that

$$f(x) < g(x) \quad \text{for all } x \in [0,1]$$

then

$$\int_0^1 f < \int_0^1 g.$$

Integrals are basically a complicated type of limit. Before working on the above problem let's try some easier limits.

Suppose $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are continuous and $f(x) < g(x)$ for all $x \in \mathbb{R}$. Prove or disprove the claim that

$$\lim_{x \to \infty} f(x) < \lim_{x \to \infty} g(x).$$

Here is a counter example

$$f(x) = 0$$

$$g(x) = \frac{1}{1+x^2}$$

then $f(x) = 0 < \frac{1}{1+x^2} = g(x)$ for all $x \in \mathbb{R}$

but

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} 0 = 0$$

and

$$\lim_{x \to \infty} g(x) = \lim_{x \to 0} \frac{1}{1+x^2} = 0.$$

This isn't that much like the limit that appears in the integrals.

Recall the lower sum approximation for the Riemann integral is given by

$$h_n(x) = \sum m_i \Delta x_i$$

where $m_i = \inf\{f(x) : x \in [x_i, x_{i+1}]\}$.

Therefore let us consider the following question:

Suppose $f: [0,1] \to \mathbb{R}$ and $g: [0,1] \to \mathbb{R}$ are bounded and $f(x) < g(x)$ for all $x \in [0,1]$

Prove or disprove the claim that

$$\inf\{f(x) : x \in [0,1]\} < \inf\{g(x) : x \in [0,1]\}$$

Note that we assume $f, g$ are bounded to avoid thing like vertical asymptotes and because this is anyway a necessary condition for $f$ and $g$ to be Riemann integrable.

If you are having trouble finding a counter example or a proof try this one:

Suppose $f:[0,1) \to \mathbb{R}$ and $g:[0,1] \to \mathbb{R}$ are continuous and $f(x) < g(x)$ for all $x \in [0,1]$. Prove or disprove the claim that

$$\inf\{f(x) : x \in [0,1]\} < \inf\{g(x) : x \in [0,1]\}.$$

Proof: Since a continuous function attains its minimum on a closed bounded interval then

There is $x_1$ such that

$$f(x_1) = \min\{f(x) : x \in [0,1]\} = \inf\{f(x) : x \in [0,1]\}$$

and $x_2$ such that

$$g(x_2) = \min\{g(x) : x \in [0,1]\} = \inf\{g(x) : x \in [0,1]\}.$$

But how to compare $f(x_1)$ to $g(x_2)$?

They are at different $x$ values.

Since $f(x_1)$ is the minimum then $f(x_1) \leq f(x_2)$. Thus

$$f(x_1) \leq f(x_2) < g(x_2)$$

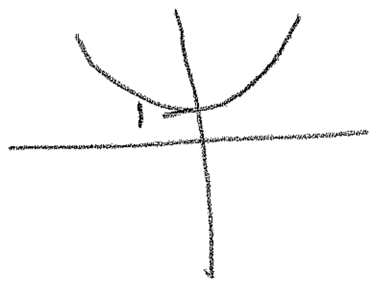
shows

$$\inf\{f(x) : x \in [0,1]\} < \inf\{g(x) : x \in [0,1]\}.$$

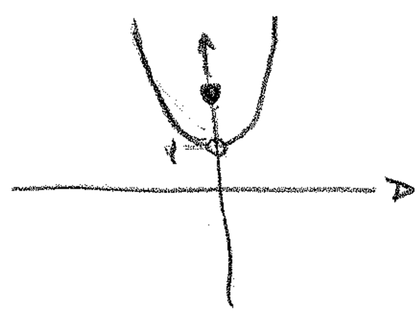
Does anyone have a proof or counter example for the case where f and g are only assumed to be bounded?

Here is a counterexample in pictures

$f(x) =$




$g(x) =$




then $f(x) < g(x)$ for all $x \in [0,1]$ but

$$\inf\{f(x) : x \in [0,1]\} = 1$$

and

$$\inf\{g(x) : x \in [0,1]\} = 1 .$$

Suppose $f(x) < g(x)$ for $x \in [0,1]$. Define

$$B_\varepsilon = \{x \in [0,1] : f(x) + \varepsilon < g(x)\}.$$

Then

$$\bigcup_{n=1}^{\infty} B_{\frac{1}{n}} = [0,1].$$

Clearly $\bigcup_{n=1}^{\infty} B_{\frac{1}{n}} \subseteq [0,1]$ since $B_{\frac{1}{n}} \subseteq [0,1]$ for all $n$.

Let $\lambda^*$ be the Lebesgue outer measure. We know that

$$\lambda^*([0,1]) = 1$$

and

$$\lambda^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \lambda^*(A_i)$$

Therefore

$$1 = \lambda^*([0,1]) = \lambda^*\left(\bigcup_{n=1}^{\infty} B_{\frac{1}{n}}\right) \leq \sum_{n=1}^{\infty} \lambda^*(B_{\frac{1}{n}})$$

means that $\lambda^*(B_{\frac{1}{n}}) > 0$ for some $n \in \mathbb{N}$, because if all the terms in the sum were zero then the sum would be zero.

One we have learned more about the Lebesgue integral we will know

$$\int \chi_A = \lambda^*(A) \text{ for all } A \in \mathcal{B}.$$

In fact, this is essentially the definition of the integral of $\chi_A$.

In this case

$$\int_0^1 g-f = \int_0^1 \frac{1}{n}\chi_{B_{\frac{1}{n}}} = \frac{1}{n}\lambda^*(B_{\frac{1}{n}}) > 0$$

implies

$$\int_0^1 f < \int_0^1 g.$$

If the result holds for Lebesgue integrable functions, then it also holds for the more restrictive condition when $f$ and $g$ are Riemann integrable.

What is the difference between $\mathcal{M}$ and $\mathcal{B}$?

What is the difference between the Lebesgue measurable functions and the Borel measurable functions?

Recall that we want to approximate $f$ by a function of the form

$$f_n = \sum_{k=-\infty}^{\infty} \frac{k}{n} \chi_{E_{n,k}}$$

where

$$E_{n,k} = f^{-1}\left(\left[\frac{k}{n}, \frac{k+1}{n}\right)\right)$$

and define the integral of $f$ in terms of the uniform limit of the $f_n$

$$\int f = \lim_{n\to\infty} \int f_n = \lim_{n\to\infty} \sum_{k=-\infty}^{\infty} \frac{k}{n} \lambda^*(E_{n,k}).$$

For this to make sense we want the measure of the disjoint union to be the sum of the measures.

In particular, we want that if

$$A_i \cap A_j = \emptyset \quad \text{for } i \neq j$$

then

$$\lambda^*\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \lambda^*(A_i)$$

In order to ensure that this kind of relation holds for the $E_{n,k}$'s we need to put a restriction on $f$.

It turns out that if we define

$$\mu : \mathcal{B} \longrightarrow \mathbb{R}^+$$

where $\mu(B) = \lambda^*(B)$.

Then for any $A_i \in \mathcal{B}$ such that

$$A_i \cap A_j = \emptyset \quad \text{for } i \neq j$$

we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

Since $\tau \subseteq \mathcal{B}$ and $\mathcal{B}$ is the smallest $\sigma$-algebra that contains $\tau$, then $\mathcal{B}$ is in some sense the smallest $\sigma$-algebra that is useful for the sort of analysis we desire. That is, which is useful for defining a measure and integration.

It is lucky that the additivity property holds for $\lambda^*$ restricted to $\mathcal{B}$.

You can also ask is there a largest $\sigma$-algebra that contains $\tau$ and consequently $\mathcal{B}$ such that

$$\lambda^*\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \lambda^*(A_i)$$

for any collection of pairwise disjoint sets in this large $\sigma$-algebra.

This requires there be a well defined largest $\sigma$-algebra. Alternatively one can simply ask is there a useful $\sigma$-algebra $\mathcal{M}$ that is bigger than $\mathcal{B}$ and for which $\lambda^*$ is still additive for disjoint sets in $\mathcal{M}$.

This is the approach taken be Lebesgue to define $\mathcal{M}$. While it is not exactly "the largest" $\sigma$-algebra for which $\lambda^*$ is additive it is large enough and significantly larger than $\mathcal{B}$. In fact

$$\mathcal{B} \sim \mathbb{R}$$

whereas

$$\mathcal{M} \sim \mathcal{P}(\mathbb{R})$$

The relationship between $\mathcal{B}$ and $\mathcal{M}$ is similar to the relationship between $\mathbb{Q}$ and $\mathbb{R}$. One is the completion of the other and the cardinality of one set is significantly larger than the other.

Thus

$$\mathcal{B} \subseteq \mathcal{M} \subseteq \mathcal{P}(\mathbb{R})$$

and $\mathcal{B} \neq \mathcal{M} \neq \mathcal{P}(\mathbb{R})$. Studying $\mathcal{M}$ is what we will be doing in sections 3.4 and 3.5.

Can you say something about the notation for problem 1? in homework 5?

$$(B_n(f)(x))^2 = ((B_n f)(x))^2$$

$$= \left( \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} \right)^2$$

(note, this is n)

and

$$B_n(f^2)(x) = (B_n f^2)(x)$$

(Note this is n)

$$= \sum_{k=0}^{n} \left(f\left(\frac{k}{n}\right)\right)^2 \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k}$$

Oops! Note that the sum appearing in the handout has a typo. The upper limit in the sum should by $n$. For a hint this is actually the first problem in Davidson and Donsig for this section.

Can you give a hint for problem 3?

<u>Hint 1</u>: If $E$ and $F$ are actually $\sigma$-algebras then $\mathcal{A}(E) = E$ and $\mathcal{A}(F) = F$.

Since the intersection of two $\sigma$-algebras is again a $\sigma$-algebra then $E \cap F = \mathcal{A}(E \cap F)$.

In this case it is clear that

$$\mathcal{A}(E) \cap \mathcal{A}(F) = \mathcal{A}(E \cap F).$$

<u>Hint 2</u>: The problem in the homework makes no assumption about $E$ and $F$ being $\sigma$-algebras. If you are looking for a counter example you better take collections of subsets $E$ and $F$ that are not both $\sigma$-algebras.

<u>Extra Credit</u>: Analyse the situation where $E$ is a $\sigma$-algebra and $F$ is not. Then is it true or false that

$$\mathcal{A}(E) \cap \mathcal{A}(F) = \mathcal{A}(E \cap F)$$

in general?

Can you give a hint for problem 3.12 in McDonald and Weiss?

I haven't worked this problem, so we could just try to work it together. Is that okay?

Note that there are lots of good problems in McDonald and Weiss that I have no assigned to turn in. These problems are also good to think about and try to work.

It should be pointed out that Alex Kumjian made a webpage with typed solutions to many of the problems in McDonald and Weiss. I have a link to these solutions on our webpage.

Problem 3.12 Define

$$A+B = \{a+b : a \in A \text{ and } b \in B\}$$

If $B$ is a Borel set and $A$ is countable then is $A+B$ a Borel set?

If $B$ is a Borel set and $A$ is open then is $A+B$ a Borel set?

Here is a lemma that might be useful:

Lemma: If $f, g \in \hat{C}$ then $f \circ g \in \hat{C}$.

Proof: By the characterization of $\hat{C}$ proved last Wednesday we have

$$\hat{C} = \mathcal{F} = \{ f : \mathbb{R} \to \mathbb{R} \text{ s.t. } f^{-1}(B) \in \mathcal{B} \text{ for every } B \in \mathcal{B} \}.$$

Let $B \in \mathcal{B}$. Since $f \in \mathcal{F}$ then $A = f^{-1}(B) \in \mathcal{B}$. Since $g \in \mathcal{F}$ then $g^{-1}(A) \in \mathcal{B}$. Therefore

$$(f \circ g)^{-1}(B) = g^{-1}(f^{-1}(B)) = g^{-1}(A) \in \mathcal{B}$$

It follows that $f \circ g \in \mathcal{F} = \hat{C}$.

Note the similarity to this characterization of $\hat{C}$ to the open set characterization of the continuous functions

$$C = \{ f : \mathbb{R} \to \mathbb{R} \text{ st. } f^{-1}(O) \in \mathcal{T} \text{ for every } O \in \mathcal{T} \}.$$

The proof that a composition of continuous functions is continuous according to this definition is the same as the proof that a composition of Borel measurable functions is Borel measurable.

Example 1

Easy case of $A$ countable

$$A = \{0\}.$$

Then

$$A+B = \{0\}+B = B \in \mathcal{B}$$

Example 2

$$A = \{0,1\}$$

Then

$$A+B = (\{0\}+B) \cup (\{1\}+B)$$
$$= B \cup (\{1\}+B)$$

Need to show $\{1\}+B$ is Borel measurable.

Hint: use the lemma to show

$$\chi_{\{1\}+B}(x) = \chi_B(x-1)$$

is measurable.

What about the case that A is open?

By the structure theorem we have

$$A = \bigcup_n I_n$$

is a countable union of disjoin open intervals, thus

$$A+B = \bigcup_n (I_n + B)$$

and we need to show $J+B$ is Borel measurable if $J$ is an open interval.

Define

$$\mathcal{A} = \{ X \subseteq \mathbb{R} : J + X \in \mathcal{B} \}.$$

Clearly $\mathcal{A}$ contains the open intervals since

$$(a,b) + (c,d) = (a+b, b+d)$$

Claim $\mathcal{A}$ is a $\sigma$-algebra

Hint: Use the fact that arbitrary unions of open sets are open to show that $\mathcal{A} = \mathcal{P}(\mathbb{R})$.