Math 713 Review/Recitation October 30, '10

Discuss what it means the intersection of collections of subsets of $\mathbb{R}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two collections of subsets of $\mathbb{R}$. The intersection of $\mathcal{A}$ and $\mathcal{B}$ is defined

$$\mathcal{A} \cap \mathcal{B} = \{ C : C \in \mathcal{A} \text{ and } C \in \mathcal{B} \}.$$

Example 1

$$\mathcal{A} = \{ [0,1], [1,2], [2,3], [3,4], [4,5], [5,6] \}$$
$$\mathcal{B} = \{ [0,1], [2,3], [4,5] \}.$$

Then

$$\mathcal{A} \cap \mathcal{B} = \{ [0,1], [2,3], [4,5] \}.$$

Note that $\mathcal{B} \subseteq \mathcal{A}$ therefore $\mathcal{A} \cap \mathcal{B} = \mathcal{A}$.

Example 2

$$\mathcal{D} = \{ (0,1), (1,2), (2,3), (3,4), (4,5), (5,6) \}.$$

Then what is $\mathcal{D} \cap \mathcal{A}$? Vote

	Vote
a) $\mathcal{D} \cap \mathcal{A} = \mathcal{D}$	2
b) $\mathcal{D} \cap \mathcal{A} = \emptyset$	4

Look at the definition. Is there any set C that is in both $\mathcal{D}$ and $\mathcal{A}$?

Discuss the hint to problem 1.

The hint is expand $B_n(f-\lambda)^2$.

This is the hint given in Davidson and Donsig for working this problem. A proof following this hint should look like the Cauchy-Schwartz inequality.

What is the Cauchy-Schwartz inequality?

Our book on page 584

$$|\langle x,y\rangle|^2 \leq \langle x,x\rangle\langle y,y\rangle$$

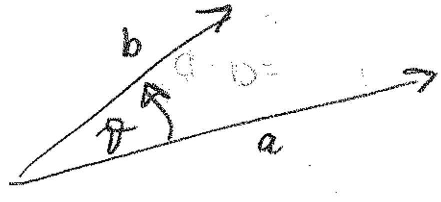
Linear algebra version

$$a\cdot b \leq \|a\|\,\|b\|$$

Calculus version

$$\int fg \leq \sqrt{\int f^2}\sqrt{\int g^2}$$

The proof in the linear algebra class usually looks like



$$a\cdot b = \|a\|\,\|b\| \cos\theta$$
$$\leq \|a\|\,\|b\|$$
since $\cos\theta \leq 1$.

There is a proof based on a hint similar to the hint for problem one that involves expanding

$$\int (f+\lambda g)^2$$

that is typical in a calculus class. This proof will work for any of the versions of the Cauchy-Schwartz inequality be cause is relies only on linearity.

<u>Proof:</u> Since $(f+\lambda g)^2 \geq 0$ then

$$0 \leq \int (f+\lambda g)^2 = \int f^2 + 2\lambda \int fg + \lambda^2 \int g^2$$

$$= A\lambda^2 + B\lambda + C$$

where $A = \int g^2$, $B = 2\int fg$ and $C = \int f^2$.

It follows that $A\lambda^2 + B\lambda + C$ is a quadratic in λ that is non-negative.

If $A=0$ then $0 \leq B\lambda + C$ implies $B=0$ in which case obviously $\int fg \leq \sqrt{\int f^2}\sqrt{\int g^2}$ since both sides are zero.

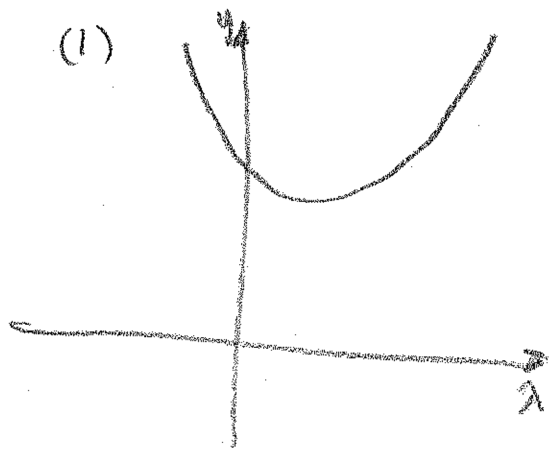
Otherwise $A \neq 0$ and since $f^2 \geq 0$ then $A > 0$. In this case the quadratic

$$A\lambda^2 + B\lambda + C$$

looks like the letter "U".

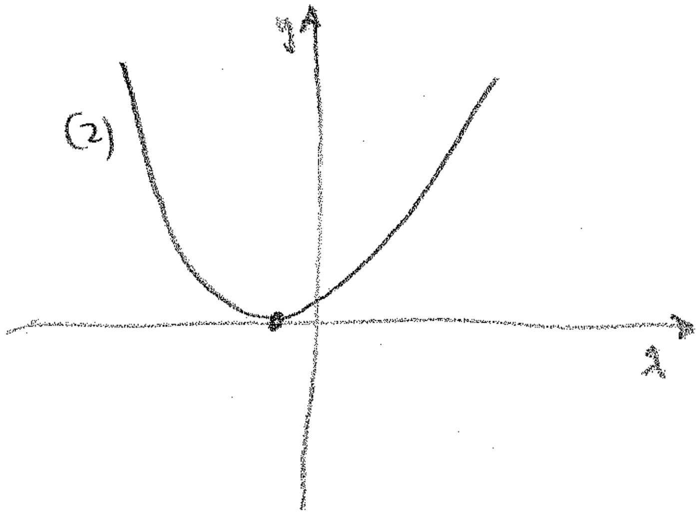
Possible graphs of $y = A\lambda^2 + B\lambda + C$ look like

(1)



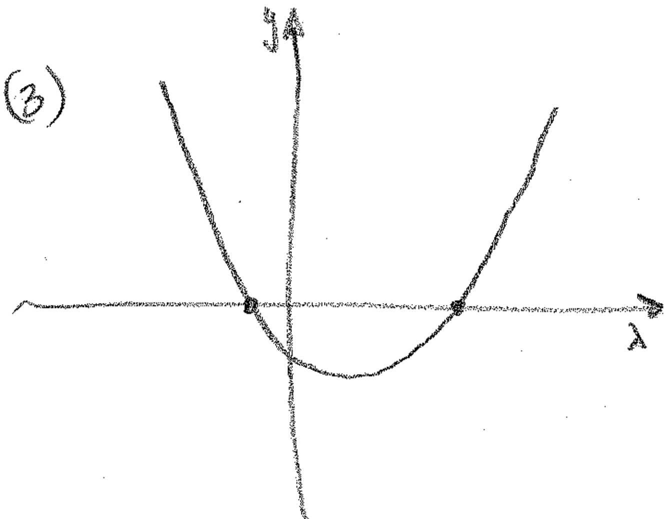
no real roots,
i.e. two complex roots

(2)



One real root
i.e. of multiplicity two.

(3)



Two real roots
i.e. part of the graph dips below the x-axis
so $A\lambda^2 + B\lambda + C < 0$ for some values of λ.

In conclusion, since

$$A\lambda^2 + B\lambda + C \geq 0$$

we must be in the situation of

(1) no real roots
(2) one real root.

The quadraditic formulae gives the roots as

$$\lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

from which we see that to avoid the case of two distinct real roots we must have

$$B^2 - 4AC \leq 0$$

That is, the discriminant must be less or equal to zoro. Substituting for A, B and C gives

$$\left(2\int fg\right)^2 \leq 4\int f^2 \int g^2$$

which implies

$$\int fg \leq \sqrt{\int f^2}\sqrt{\int g^2}.$$

The proof that

$$(B_n f)^2(x) \leq (B_n(f^2))(x) \quad \text{for } x \in [0,1]$$

can be done in a similar fashion.

Alternatively, it may be possible to use the Cauchy-Schwartz inequality directly to obtain the result.

<u>Discuss</u> any hints for showing that $\mathbb{P}(\mathbb{R})$ is not set equivalent to $\mathbb{R}$.

This is a standard result in the discussions on cardinal numbers that appear in most set theory books and some analysis books. However, the result does not seem to be in our book, so it is extra credit.

The proof is rather tricky but also short. It is a good exercise just to look up the proof online or in the library and write it down.

The proof is sometimes termed a diagonalization argument, though it appears differently than the one used to show $\mathbb{N}$ is not set equivalent to $\mathbb{R}$.

Any ideas? Proof by contradiction.

So lets assume there is a bijective function

$$f: \mathbb{R} \to \mathcal{P}(\mathbb{R}).$$

My recollection is to define a set like

$$D = \{x \in \mathbb{R} : x \in f(x)\}.$$

and try to obtain a contradiction someway.

Since $D \subseteq \mathbb{R}$ and f is onto then there must be some $x_0 \in \mathbb{R}$ such that $f(x_0) = D$.

Similarly $D^c \subseteq \mathbb{R}$ and so there is $x_1 \in \mathbb{R}$ such that $f(x_1) = D^c$.

Now the contradiction comes from trying to decide whether x_0 or x_1 are themselves members of D or members of D^c.

Fill in the details...