Math 713 Summary November 1, 2010

Last week we defined

$$\mathcal{M}=\{E\subseteq\mathbb{R} : \lambda^*(W)=\lambda^*(W\cap E)+\lambda^*(W\cap E^c) \text{ for all } W\subseteq\mathbb{R}\}.$$

and proved $\mathcal{M}$ is an algebra of sets.

We want to show

(i) $\mathcal{M}$ is a σ-algebra

(ii) $\mathcal{B}\subseteq\mathcal{M}$

(iii) λ^* is additive under countable disjoint unions of sets in $\mathcal{M}$.

Let's start with a lemma that is related to (iii).

<u>Lemma 1.</u> Let $A_i\in\mathcal{M}$ for $i=1,2,\dots,n$ be a disjoint collection of sets and $W\subseteq\mathbb{R}$. Then

$$\lambda^*\left(W\cap\bigcup_{i=1}^{n}A_i\right)=\sum_{i=1}^{n}\lambda^*(W\cap A_i).$$

<u>Proof</u>; We shall show this in the case $n=2$, since the general case follows easily from this using induction.

Note that since $\mathcal{M}$ is an algebra then $A_1\cup A_2\in\mathcal{M}$.

Now since $A_1 \in \mathcal{M}$ then

$$\lambda^*(W) = \lambda^*(W \cap A_1) + \lambda^*(W \cap A_1^c)$$

for any $W \subseteq \mathbb{R}$.

In particular, replacing W by $W \cap (A_1 \cup A_2)$ we obtain

$$\lambda^*(W \cap (A_1 \cup A_2)) = \lambda^*(W \cap (A_1 \cup A_2) \cap A_1) + \lambda^*(W \cap (A_1 \cup A_2) \cap A_1^c).$$

Since $A_1 \cap A_2 = \emptyset$ then $A_2 \subseteq A_1^c$. It follows that

$$W \cap (A_1 \cup A_2) \cap A_1^c = W \cap A_2.$$

Also

$$W \cap (A_1 \cup A_2) \cap A_1 = W \cap A_1$$

Therefore

$$\lambda^*(W \cap (A_1 \cup A_2)) = \lambda^*(W \cap A_1) + \lambda^*(W \cap A_2)$$

which finishes the proof for $n=2$. As mentioned earlier the general result now follows from induction

Taking $W = \mathbb{R}$ in the previous lemma leads imediatly to

Corollary: Let $A_i \in \mathcal{M}$ for $i = 1, 2, \dots, n$ be a disjoint collection of sets. Then

$$\lambda^*\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \lambda^*(A_i).$$

Therefore we have shown that λ^* restricted to sets of $\mathcal{M}$ is additive for finite disjoint unions of sets in $\mathcal{M}$.

We are left with showing that countably infinite unions behave just as well.

Lemma 2. Let $A_i \in \mathcal{M}$ for $i \in \mathbb{N}$ be a pairwise disjoint collection of sets and $W \subseteq \mathbb{R}$. Then

$$\lambda^*\left(W \cap \bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \lambda^*(W \cap A_i).$$

Proof: Since $\bigcup_{i=1}^{\infty} A_i \supseteq \bigcup_{i=1}^{n} A_i$ we have by proposition 3.1 in the text that

$$\lambda^*(W \cap \bigcup_{i=1}^{\infty} A_i) \geq \lambda^*(W \cap \bigcup_{i=1}^{n} A_i) \quad \text{for every } n \in \mathbb{N}.$$

Therefore by the previous lemma

$$\lambda^*(W \cap \bigcup_{i=1}^{\infty} A_i) \geq \sum_{i=1}^{n} \lambda^*(W \cap A_i) \text{ for every } n \in \mathbb{N}.$$

Taking limits we obtain that

$$\lambda^*(W \cap \bigcup_{i=1}^{\infty} A_i) \geq \sum_{i=1}^{\infty} \lambda^*(W \cap A_i).$$

Since proposition 3.1 subadditivity implies

$$\lambda^*(W \cap \bigcup_{i=1}^{\infty} A_i) = \lambda^*(\bigcup_{i=1}^{\infty}(W \cap A_i)) \leq \sum_{i=1}^{\infty} \lambda^*(W \cap A_i)$$

we conclude that

$$\lambda^*(W \cap \bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \lambda^*(W \cap A_i).$$

thus proving the lemma.

Corollary: Let $A_i \in \mathcal{M}$ for $i \in \mathbb{N}$ be a pairwise disjoint collection of sets. Then

$$\lambda^*\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \lambda^*(A_i).$$

This corollary is simply the result of taking $W = \mathbb{R}$ in the previous lemma.

We are now ready to show $\mathcal{M}$ is a σ-algebra.

Since $\mathcal{M}$ is already known to be an algebra, it is enough to show $\mathcal{M}$ is a monotone class.

Let $E_i \in \mathcal{M}$ such that $E_1 \subseteq E_2 \subseteq \cdots$.

Define $A_1 = E_1$ and $A_{i+1} = E_{i+1} \setminus E_i$.

Then A_i are pairwise disjoint and

$$E_n = \bigcup_{i=1}^{n} E_i = \bigcup_{i=1}^{n} A_i$$

for every $n \in \mathbb{N}$.

Since $\bigcup_{i=1}^{\infty} A_i \supseteq \bigcup_{i=1}^{n} A_i = E_n$ then

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c \subseteq E_n^c .$$

Moreover since $E_n \in \mathcal{M}$ we have

$$\lambda^*(W) = \lambda^*(W \cap E_n) + \lambda^*(W \cap E_n^c)$$

$$\geq \lambda^*\left(W \cap \bigcup_{i=1}^{n} A_i\right) + \lambda^*\left(W \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right).$$

Since the A_i are disjoint lemma 1 implies

$$= \sum_{i=1}^{n} \lambda^*(W \cap A_i) + \lambda^*\left(W \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right)$$

As this holds for all $n \in \mathbb{N}$, taking limits and applying lemma 2 yields

$$\lambda^*(W) \geq \sum_{i=1}^{\infty} \lambda^*(W \cap A_i) + \lambda^*\left(W \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right)$$

$$= \lambda^*\left(W \cap \bigcup_{i=1}^{\infty} A_i\right) + \lambda^*\left(W \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right).$$

Now subadditivity in proposition 3.1 gives

$$\lambda^*(W) = \lambda^*\left(\left(W \cap \bigcup_{i=1}^{\infty} A_i\right) \cup \left(W \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right)\right)$$

$$\leq \lambda^*\left(W \cap \bigcup_{i=1}^{\infty} A_i\right) + \lambda^*\left(W \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right)$$

Therefore

$$\lambda^*(W) = \lambda^*\left(W \cap \bigcup_{i=1}^{\infty} A_i\right) + \lambda^*\left(W \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right),$$

which shows $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} E_i \in M$.

Therefore M is closed under monotone increasing unions of subsets,

Let $E_i \in M$ and $E_1 \supseteq E_2 \supseteq \cdots$ Then $E_1^c \subseteq E_2^c \subseteq \cdots$ and so $\bigcup_{i=1}^{\infty} E_i^c \in M$.
Since M is closed under compements, then DeMorgans law implies

$$\bigcap_{i=1}^{\infty} E_i = \left(\bigcup_{i=1}^{\infty} E_i^c\right)^c \in M.$$

This shows that $\mathcal{M}$ is a monotone class. It follows that $\mathcal{M}$ is a σ-algebra.

We now have that

(i) $\mathcal{M}$ is a σ-algebra

(iii) λ^* is additive on $\mathcal{M}$.

It remains to show $\mathcal{B} \subseteq \mathcal{M}$.

Note that since $\mathcal{B}$ is the smallest σ-algebra that contains

$$\tau = \{ O \subseteq \mathbb{R} : O \text{ is open} \}$$

this implies that $\mathcal{M}$ contains enough sets that allow us to do the analysis needed to form a theory of integration.

<u>Question</u>: Is λ^* restricted to the σ-algebra $\mathcal{E} = \{\emptyset, \mathbb{R}\}$ additive? <u>YES</u>

After we show $\mathcal{B} \subseteq \mathcal{M}$ we will have

$$\mathcal{E} \subseteq \mathcal{B} \subseteq \mathcal{M} \subseteq \mathcal{P}(\mathbb{R})$$

$\mathcal{P}(\mathbb{R})$ is too big a σ-algebra because λ^* is not additive for disjoint subsets of $\mathcal{P}(\mathbb{R})$.

$\mathcal{E}$ is too small, since no interesting theory of integration could be based on λ^* restricted to this σ-algebra.

$\mathcal{M}$ is defined in a natural way that allows us to prove λ^* is additive on $\mathcal{M}$ and turn out to contain $\mathcal{B}$ so it is large enough to develop an interesting theory of integration.

Of course we still need to show $\mathcal{B} \subseteq \mathcal{M}$. This proof is given of pages 114 to 115 in the text. Please read pages 112 to 113 in preparation for the proof that $\mathcal{B} \subseteq \mathcal{M}$.

At the beginning of the course we skipped the section on the Cantor set.

Now we will cover that section.

First, a few observations:

<u>Proposition</u>: If $\lambda^*(A)=0$ then $A\in\mathcal{M}$.

<u>Proof</u>: $W\cap A\subseteq A$ so $\lambda^*(W\cap A)=0$. Therefore

$$\lambda^*(W\cap A)+\lambda^*(W\cap A^c)=\lambda^*(W\cap A^c)\leq\lambda^*(W).$$

Since the reverse inequality is obvious from subadditivity we obtain that

$$\lambda^*(W\cap A)+\lambda^*(W\cap A^c)=\lambda^*(W)$$

and therefore $A\in\mathcal{M}$.

An obvious consequence of this is that $\mathbb{Q}=\{q_1,q_2,q_3,\dots\}=\bigcup_{n=1}^{\infty}\{q_n\}\in\mathcal{M}$ since it's the countable union of the sets $\{q_n\}\in\mathcal{M}$ where $\lambda^*(\{q_n\})=0$.

Moreover

$$\lambda^*(\mathbb{Q}) = \sum_{n=1}^{\infty} \lambda^*(\{q_n\}) = \sum_{n=1}^{\infty} 0$$

and therefore $\mathbb{Q}$ has Lebesgue outer measure equal to zero.

Similarly any countable set A has $\lambda^*(A)=0$. Is the converse true?

In particular, is the claim that $\lambda^*(A)=0$ if and only if A is countable true or false?

Let I be an interval $I=(a,b)$.
Then
$$\lambda^*(I) = \lambda^*((b-a)) = b-a.$$

Let $f:(a,b) \to (\frac{a}{2}, \frac{b}{2})$ be given by $f(x)=\frac{x}{2}$.
Then f is a bijection and so $I \sim f(I)$.
Also
$$\lambda^*(f(I)) = \lambda^*\left(\left(\frac{a}{2}, \frac{b}{2}\right)\right) = \frac{b-a}{2}$$

Thus (a, b) and $(\frac{a}{2}, \frac{b}{2})$ are two sets that are equivalent in the sense of set equivalence, which means they contain the same number of points, and the Lebesgue measure of one of them is $\frac{1}{2}$ the Lebesgue measure of the other.

Is it possible to stretch and squeeze the unit interval in such a way that the Lebesgue measure of that interval it not only smaller but exactly zero?

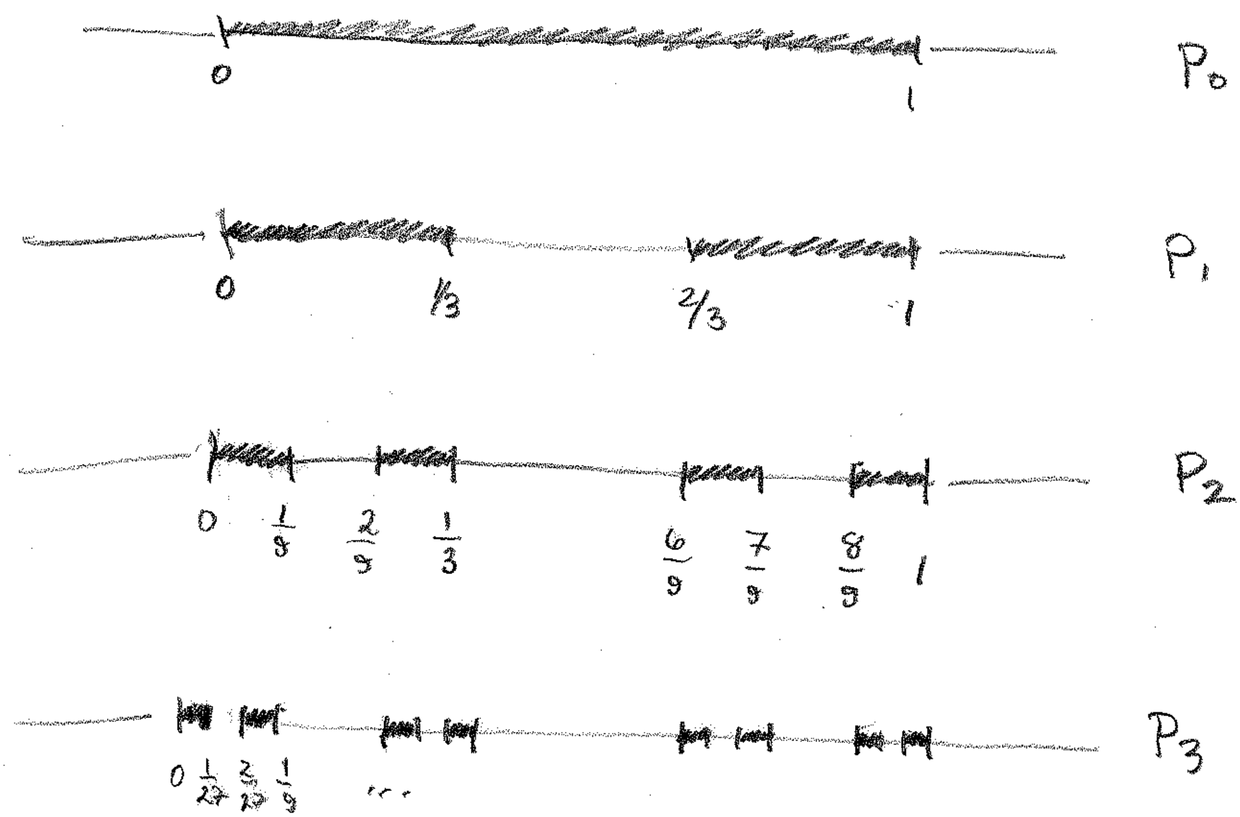
YES.

This gives us an example of an set $P \subseteq \mathbb{R}$ that is uncountable but for which

$$\lambda^*(P) = 0.$$

Thus, the claim on the previous page is false.

The set P is called the Cantor set and it is defined as $P = \bigcap_{n=1}^{\infty} P_n$ where P_n are the closed sets given as



Consider the complement of P_n.

$$E_0 = [0,1] \setminus P_0$$

$$E_1 = [0,1] \setminus P_1 = \left(\tfrac{1}{3}, \tfrac{2}{3}\right)$$

$$E_2 = [0,1] \setminus P_2 = \left(\tfrac{1}{3}, \tfrac{2}{3}\right) \cup \left(\tfrac{1}{9}, \tfrac{2}{9}\right) \cup \left(\tfrac{7}{9}, \tfrac{8}{9}\right)$$

...

Define $A_1 = G_1$ and $A_{n+1} = G_{n+1} \setminus G_n$. Then the A_i's are disjoint and

A_1 is 1 interval of length $\frac{1}{3}$

A_2 is 2 —— // —— $\frac{1}{9}$

A_3 is 4 —— // —— $\frac{1}{27}$

$\vdots$

Since each A_i is open then $A_i \in \mathcal{B}$. Assuming we have already shown that $\mathcal{B} \subseteq \mathcal{M}$, it then follows that $A_i \in \mathcal{M}$ and

$$\lambda^*([0,1] \cap P^c) = \lambda^*\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \lambda^*(A_i)$$

$$= \sum_{i=1}^{\infty} 2^{i-1} \frac{1}{3^i} = \frac{1}{2} \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^i$$

$$= \frac{1}{2}\left(\frac{\frac{2}{3}}{1-\frac{2}{3}}\right) = 1.$$

Therefore $P \in \mathcal{M}$ and

$$\lambda^*([0,1]) = \lambda^*([0,1] \cap P) + \lambda^*([0,1] \cap P^c)$$
$$= \lambda^*(P) + 1$$

implies

$$\lambda^*(P) = \lambda^*([0,1]) - 1 = 0 \ ,$$

We will show that $P \sim [0,1]$ next time.