713 Recitation / Review

Nov 5–6, 2010

Discuss the construction on page 120 and the exercise 3.30 from the text.

3.30. Let E_n be any sequence of subsets of $\mathbb{R}$. Define $A_1 = E_1$, $A_2 = E_2 \setminus E_1$, $A_3 = E_3 \setminus (E_1 \cup E_2)$ and in general

$$A_n = E_n \setminus \bigcup_{k=1}^{n-1} E_k \quad \text{for } n \in \mathbb{N}.$$

Prove that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} E_n$.

<u>Claim</u> $A_i \cap A_j = \emptyset$ for $i \neq j$.

<u>proof</u> For definiteness assume $i < j$.

<u>Case</u> $i = 1$. Then $j \geq 2$ and

$$A_1 \cap A_j = E_1 \cap \left(E_j \setminus \bigcup_{k=1}^{j-1} E_k\right).$$

$$= E_1 \cap E_j \cap \left(\bigcup_{k=1}^{j-1} E_k\right)^c$$

$$= E_1 \cap E_j \cap \bigcap_{k=1}^{j-1} E_k^c$$

$$\subseteq E_1 \cap E_j \cap E_1^c = \emptyset$$

<u>Case</u> $i \geq 2$. Then

$$A_i \cap A_j = \left(E_i \setminus \bigcup_{k=1}^{i-1} E_k\right) \cap \left(E_j \setminus \bigcup_{k=1}^{j-1} E_k\right)$$

$$A_i \cap A_j \subseteq E_i \cap \left(E_j \setminus \bigcup_{k=1}^{j-1} E_k\right)$$

$$= E_i \cap E_j \cap \left(\bigcup_{k=1}^{j-1} E_k\right)^c$$

$$= E_i \cap E_j \cap \bigcap_{k=1}^{j-1} E_k^c$$

$$\subseteq E_i \cap E_j \cap E_i^c = \emptyset \quad \text{since } j > i.$$

<u>Claim</u> $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} E_n$

"$\subseteq$" Since $A_1 = E_1$ and $A_n = E_n \setminus \bigcup_{k=1}^{n-1} E_k \subseteq E_n$ then $A_n \subseteq E_n$ for all $n \in \mathbb{N}$.

It follows that

$$\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} E_n$$

"$\supseteq$" Suppose $x \in \bigcup_{n=1}^{\infty} E_n$. Then $x \in E_n$ for some $n \in \mathbb{N}$. Define $J = \{n \in \mathbb{N} : x \in E_n\}$ and let $j_0 = \inf J$. Since $J \subseteq \mathbb{N}$ and $J \neq \emptyset$ then this infimum exists and is finite.

<u>Claim</u> $x \in A_{j_0}$

Suppose, for contradiction, that $x \notin A_{j_0}$. Then since $x \in E_{j_0}$ we must also have $x \in \bigcup_{k=1}^{j_0-1} E_k$.

This means $x \in E_{k_0}$ for some $k_0 \in \{1, 2, \dots, j_0-1\}$.

Since $x \in E_{k_0}$ then $k_0 \in J$. It follows that

$$j_0 = \inf J \leq k_0 \leq j_0 - 1 \quad \text{since } k_0 \in \{1, 2, \dots, j_0-1\}.$$

This is a contradiction.

Therefore $x \in A_{j_0}$.

Since $x \in A_{j_0}$ then $x \in \bigcup_{j=1}^{\infty} A_j$.

This shows that

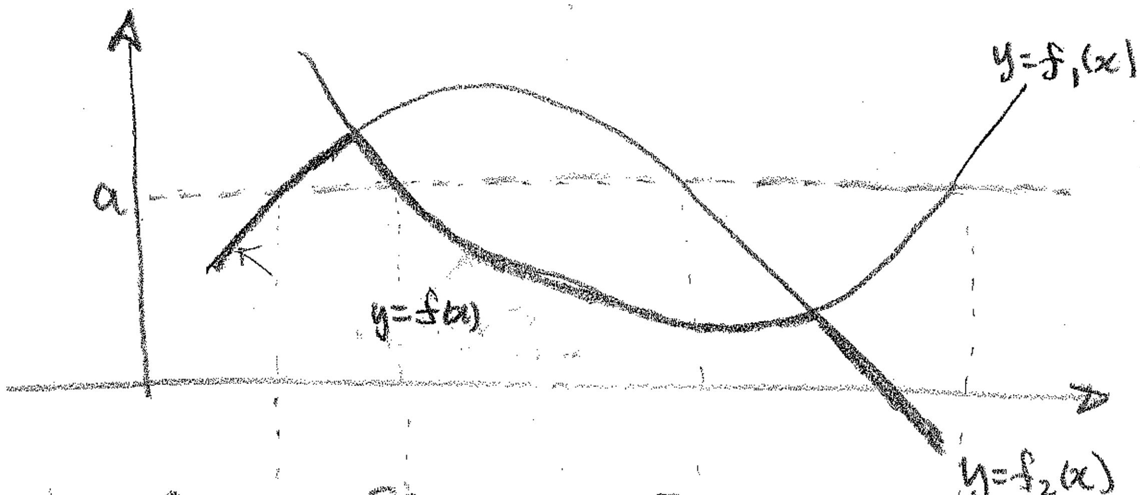
$$\bigcup_{n=1}^{\infty} A_n \supseteq \bigcup_{n=1}^{\infty} E_n.$$

Therefore

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} E_n.$$

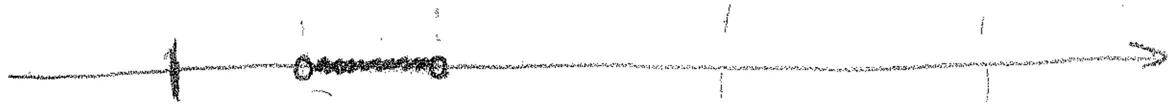
Discussion about the homework. problem 1.
What's going on?

Here is an illustration.

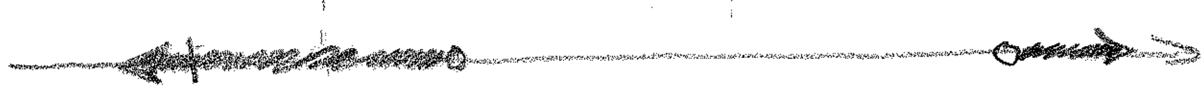


where $f(x)=\inf\{f_1(x), f_2(x)\}$.

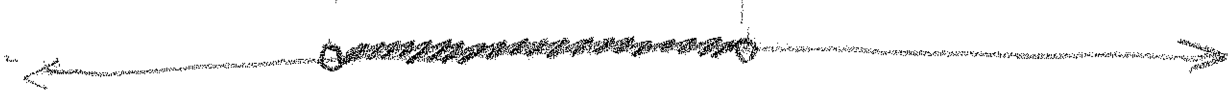
Then $f^{-1}((a,\infty))$ is



and $f_1^{-1}((a,\infty))$ is



and $f_2^{-1}((a,\infty))$ is



In this example

$$f^{-1}((a,\infty)) = f_1^{-1}((a,\infty)) \cap f_2^{-1}((a,\infty)),$$

So we may have a partial result:

If the functions are continuous and there are only two of them then the result is probably true. It is worth trying to write a complete proof of this result trying the general problem.

After writing the proof for the case of two continuous functions you could try to generalize to two functions which aren't continuous or to countably infinite collection of continuous functions.

By proving the result for two continuous functions it may help find a proof of the general result, or it may help in how to construct a counterexample for the general result.

Discussion of problem 2 on homework #6.

Let $E \subseteq \mathbb{R}$. Suppose for every $\varepsilon > 0$ there is $F \in \mathcal{B}$ such that $F \subseteq E$ and $\lambda^*(E \setminus F) < \varepsilon$. Prove or disprove that $E \in \mathcal{B}$.

This is a modification of a problem appearing in a book which says...

Let $E \subseteq \mathbb{R}$. Suppose for every $\varepsilon > 0$ there is $F \in \mathcal{M}$ such that $F \subseteq E$ and $\lambda^*(E \setminus F) < \varepsilon$. Prove that $E \in \mathcal{M}$.

So, unless I'm remembering wrong, the result is true for $\mathcal{M}$, but I don't know about $\mathcal{B}$.

A possible approach would be to examine a more general statement to see if there are any special properties of $\mathcal{M}$ which might be needed for this result.

<u>General statement</u>:

Let $\mathcal{A}$ be a σ-algebra of subsets of $\mathbb{R}$ and $E \subseteq \mathbb{R}$. Suppose for every $\varepsilon > 0$ there is $F \in \mathcal{A}$ such that $F \subseteq E$ and $\lambda^*(E \setminus F) < \varepsilon$. Prove that $E \in \mathcal{A}$.

Let's try the simplest σ-algebra

$$\mathcal{A} = \{\emptyset, \mathbb{R}\}.$$

Let $E = \mathbb{Q}$ and $F = \emptyset$. Then $E \subseteq \mathbb{R}$ and $F \in \mathcal{A}$. Moreover, for any $\varepsilon > 0$ we have

$$\lambda^*(E \setminus F) = \lambda^*(\mathbb{Q}) = 0 < \varepsilon.$$

However $\mathbb{Q} \notin \mathcal{A}$.

Therefore, the result is not true for $\mathcal{A} = \{\emptyset, \mathbb{R}\}$.

<u>Question</u>, is it true for $\mathcal{A} = \mathcal{B}$?

Since it wasn't true for $\{\emptyset, \mathbb{R}\}$ I'm thinking probably not for $\mathcal{B}$ either, but this is likely much more difficult to prove.

<u>Recall the theorem</u>: If $\lambda^*(A) = 0$ then $A \in \mathcal{M}$.

Is there a set A ~~such that~~ $\lambda^*(A) = 0$ but $A \notin \mathcal{B}$?

This might help with answering the problem.

Let's return to last week's problem, exercise 3.50, to see if the set described there satisfies the requirement $\lambda^*(A)=0$ but $A \notin \mathcal{B}$.

Exercise 3.50 is essentially the outline of the proof of a theorem with the details left out.

We have already worked parts (a), (b) and (c) last week. What remains is parts (d)-(g).

(d) Define $f: [0,1] \to \mathbb{R}$ by $f(x) = x + \Psi(x)$ where Ψ denotes the Cantor function. Then f is a strictly increasing function that maps $[0,1]$ onto $[0,2]$.

Since Ψ is non-decreasing then $x+\Psi(x)$ is strictly increasing. In particular if $x_1 < x_2$ then

$$f(x_1) = x_1 + \Psi(x_1) \leq x_1 + \Psi(x_2) < x_2 + \Psi(x_2) = f(x_2)$$

So the function f is strictly increasing.

Since f is the sum of continuous functions it is continuous. Since

$$f(0) = 0 + \psi(0) = 0$$

and

$$f(1) = 1 + \psi(1) = 1 + 1 = 2$$

Then the intermediate value theorem implies that f maps $[0,1]$ onto $[0,2]$.

(e) The function $g = f^{-1}$ is continuous and hence Borel measurable.

Since f is strictly increasing it is 1-to-1 and so g is well defined.

Let $x_1, x_2 \in [0,1]$ with $x_1 < x_2$. Then

$$f(x_2) - f(x_1) = x_2 + \psi(x_2) - x_1 - \psi(x_1) \geq x_2 - x_1$$

Implies that

$$|f(x_2) - f(x_1)| \geq |x_2 - x_1|$$

for all $x_1, x_2 \in [0,1]$.

Now let $y_1, y_2 \in \mathbb{R}$.

Define $x_1 = g(y_1)$ and $x_2 = g(y_2)$. Then

$$|g(y_2) - g(y_1)| = |x_2 - x_1| \le |f(x_2) - f(x_1)|$$
$$= |g^{-1}(x_2) - g^{-1}(x_1)| = |y_2 - y_1|$$

This implies that g is uniformly continuous.
Therefore g is continuous and therefore $g \in \hat{C}$.

(f) f maps the Cantor set onto a set A with $\lambda^*(A) = 1$.

Let $A = f(P)$ where P is the Cantor set.

Since $A = g^{-1}(P)$ and $P \in \mathcal{B}$ then $A \in \mathcal{B}$.

Since $\mathcal{B} \subseteq \mathcal{M}$ then $\lambda^*(A) = \lambda(A)$ where λ is the restriction of λ^* to the domain $\mathcal{M}$.

We will skip (f) and come back to it next week. Let's assume $\lambda(A) = 0$ has been verified and continue...

(g) Let $E \subseteq A$ with $E \notin \mathcal{M}$. Then $f^{-1}(E) \in \mathcal{M}$ but $f^{-1}(E) \notin \mathcal{B}$.

Since $\lambda^*(A) > 0$ there is $E \subseteq A$ such that $E \notin \mathcal{M}$. Recalling the proof in part (c) we know that $E = A \cap (S + r)$ for some $r \in \mathbb{Q}$ where S is the set defined as

$$x \sim y \quad \text{if and only if} \quad x - y \in \mathbb{Q}$$

$$E_x = \{ y \in \mathbb{R} : y \sim x \}$$

$$\mathcal{C} = \{ E_x : x \in \mathbb{R} \},$$

$$w : \mathcal{C} \to \mathbb{R} \quad \text{st} \quad w(E_x) \in E_x \text{ for } x \in \mathbb{R},$$

$$T = \{ w(E_x) : x \in \mathbb{R} \}$$

$$S = \{ z - [\![z]\!] : z \in T \}.$$

with the property $S \notin \mathcal{M}$.

Now $f^{-1}(E) \in \mathcal{M}$ because

$$f^{-1}(E) \subseteq f^{-1}(A) = f^{-1}(f(P)) = P$$

and therefore $\lambda^*(f^{-1}(E)) \leq \lambda^*(P) = 0$.

This implies $f^{-1}(E) \in \mathcal{M}$.

Claim $f^{-1}(E) \notin \mathcal{B}$.

Suppose $f^{-1}(E) \in \mathcal{B}$. Then since $g \in \hat{C}$ we would have $g^{-1}(f^{-1}(E)) \in \mathcal{B}$.

But

$$g^{-1}(f^{-1}(E)) = f(f^{-1}(E)) = E \notin \mathcal{M}$$

and since $\mathcal{B} \subseteq \mathcal{M}$ this is a contradiction.

Therefore $f^{-1}(E) \notin \mathcal{B}$.

Please look at part (f) of this problem and try to find a proof for this in time for next week's review session.

Note: This is a standard example and construction that will appear with details worked out in other analysis books.