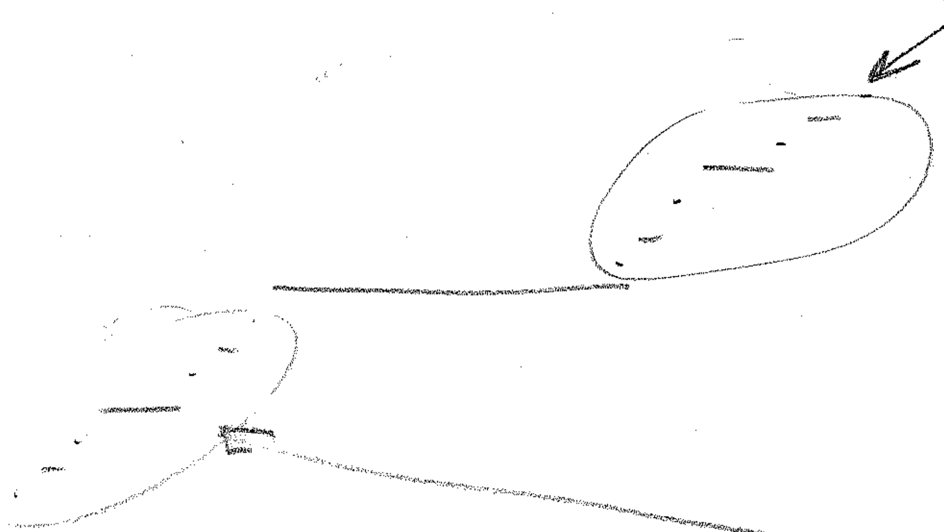
Math 713 Summary Nov 8, 2010

Last time we used maple to draw a graph of the Cantor function. From the graph it appears to be a non-decreasing function which might also be continuous.

The Cantor function is an example of a self similar fractal. See how the part here



is exactly the same as the part here

Let's prove that the function is non-decreasing. First we consider some properties of base n expansions which are easiest to understand in the most familiar setting of decimals.

True or false ...

Let $x = 0.d_1d_2d_3\ldots = \sum_{n=1}^{\infty} \frac{d_n}{10^n}$

$y = 0.e_1e_2e_3\ldots = \sum_{n=1}^{\infty} \frac{e_n}{10^n}$

Prove or disprove the claim that

$x \leq y$ implies $d_1 \leq e_1$.

For example

$.345 \leq .428$

and $3 \leq 4$.

Also

$.347 \leq .349$

and $3 \leq 3$.

So this claim looks true. Does anyone have a counter example? Let's vote.

Claim $x \leq y$ implies $d_1 \leq e_1$

True	False
12	9

Those who voted false, do you have an example?

$$.5\overline{00} \leq .4\overline{99}$$

however $5 > 4$.

Recall the Cantor set is defined as

$$P = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{3^n} : a_n \in \{0,2\} \right\}$$

If $x, y \in P$ and

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \quad \text{where } a_n \in \{0,2\}$$

$$y = \sum_{n=1}^{\infty} \frac{e_n}{3^n} \quad \text{where } e_n \in \{0,2\}$$

Is it true that $x \leq y$ implies $a_1 \leq e_1$.

In base 3 the 2s play the same role as the 9s play in base 10. Can you construct a counter example? Can you prove the result?

Proof: For contradiction, suppose $a_1 > e_1$. then the only possibility is that $a_1 = 2$ and $e_1 = 0$. Then

$$x - y = \sum_{n=1}^{\infty} \frac{a_n}{3^n} - \sum_{n=1}^{\infty} \frac{e_n}{3^n} = \frac{2}{3} + \sum_{n=2}^{\infty} \frac{a_n}{3^n} - \sum_{n=2}^{\infty} \frac{e_n}{3^n}$$

$$= \frac{2}{3} + \sum_{n=2}^{\infty} \frac{a_n - e_n}{3^n} \geq \frac{2}{3} - \sum_{n=2}^{\infty} \frac{2}{3^n}$$

$$= \frac{2}{3} - 2\,\frac{\frac{1}{3^2}}{1 - \frac{1}{3}} = \frac{2}{3} - 2\frac{\left(\frac{1}{3^2}\right)}{\left(\frac{2}{3}\right)}.$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3} > 0$$

However $x \leq y$ implies $x - y \leq 0$. This is a contradiction. Therefore $a_1 \leq e_1$.

We generalize this result to prove $f: P \to [0,1]$ defined last week is non-decreasing.

Recall

$$f\left(\sum_{n=1}^{\infty} \frac{a_n}{3^n}\right) = \sum_{n=1}^{\infty} \frac{a_n/2}{2^n} \quad \text{where } a_n \in \{0,2\}.$$

Following a similar argument as before we know that

$$x < y$$

implies there is some $k_0 \in \mathbb{N}$ such that

$$a_k = e_k \text{ for } k < k_0$$

and $a_{k_0} < e_{k_0}$.

In particular, if this were not true there would be either $k_0 \in \mathbb{N}$ such that

$$a_k = e_k \text{ for } k < k_0 \text{ and } a_{k_0} > e_{k_0}$$

or that

$$a_k = e_k \text{ for } k \in \mathbb{N}.$$

First case can't happen because of an argument similar to what is on the previous page. The second case would implies $x=y$, so it can't happen either.

Therefore if $x<y$ we have $k_0 \in \mathbb{N}$

$$a_k = e_k \text{ for } k < k_0 \text{ and } a_{k_0} < e_{k_0}.$$

Thus

$$\frac{a_k}{2} \leq \frac{e_k}{2} \text{ for } k < k_0 \text{ and } \frac{a_{k_0}}{2} < \frac{e_{k_0}}{2}.$$

It easily follows that

$$\sum_{k=1}^{\infty} \frac{a_k/2}{2^k} \leq \sum_{k=1}^{\infty} \frac{e_k/2}{2^k}.$$

Note the phrase "easily follows" means that some details have been omitted that should be possible to figure out. Please do try to fill in the missing details here.

Except for these missing details we have shown that $x<y$ implies $f(x) \leq f(y)$. Therefore f is monotone non-decreasing.

<u>Question</u>, is it, infact, true that

$$x<y \text{ implies } f(x)<f(y)?$$

To show a function is non-decreasing we only need to show

$$x < y \text{ implies } f(x) \le f(y)$$

However could it be that $f : P \to [0,1]$ is actually strictly increasing?

No!

Can you find an example?

$$.020220\overline{22}_{\text{base } 3} \to .010110\overline{11}_{\text{base } 2}$$

$$.020222\overline{00}_{\text{base } 3} \to .010111\overline{00}_{\text{base } 2}$$

and

$$.010110\overline{11}_{\text{base } 2} = .010111\overline{00}_{\text{base } 2}$$

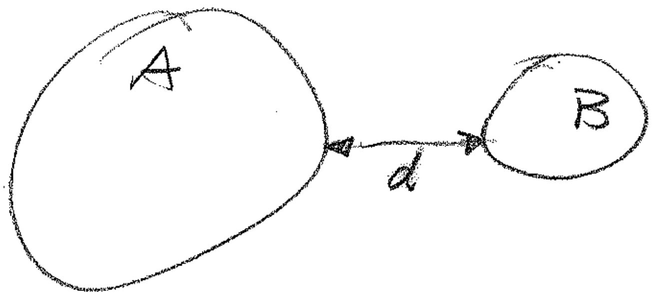
So here is an example of $x < y$ where $f(x) = f(y)$.

Let's take a break, and when we return we will discuss the proof that $\mathcal{B} \subseteq \mathcal{M}$.

Define the distance between two sets

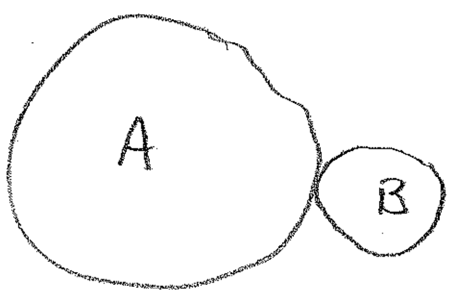
$$d(A,B) = \inf\{|a-b| : a \in A \text{ and } b \in B\}$$

Thus, in this picture



d is the distance between sets A and B.

If the distance between A and B is 0 the look like



I.e. they touch. Note that they might touch but not actually intersect. If they intersect the distance between them is also 0.

If x and y are two numbers and the distance between them is zero, then $x=y$.

If A and B are two sets and the distance between them is zero they may or may not be equal.

<u>Question</u>: If $A \approx B$ means $d(A,B)=0$ then is $\approx$ an equivalence relation?

If you are reading these notes and attended class on Monday you will notice that I did not ask this question on Monday.

Please send an email to

ejolson@unr.edu

with your vote in the subject line as

subject: 713 vote it is an equivalence relation

or

subject: 713 vote it is NOT an equivalence relation.

For an ε amount of extra credit.

Lemma: If $A, B \subseteq \mathbb{R}$ and $d(A,B) > 0$ then $\lambda^*(A \cup B) = \lambda^*(A) \cup \lambda^*(B)$.

Note that $d(A,B) > 0$ implies $A \cap B = \emptyset$ because if $A \cap B \neq \emptyset$ then $d(A,B) = 0$.

Before proving this lemma lets provide a slightly modified definition of the Lebesgue outer measure:

$$\lambda^*_\delta(A) = \inf\left\{ \sum \ell(I_n) : \begin{array}{l} I_n \text{ are open intervals} \\ \text{such that } \ell(I_n) < \delta \\ \text{and } A \subseteq \bigcup I_n \end{array} \right\}$$

again the sum and union appearing in the above definition may be finite or countably infinite. This is why the book doesn't put in the limits for the index n. The original definition of outer measure is

$$\lambda^*(A) = \inf\left\{ \sum \ell(I_n) : \begin{array}{l} I_n \text{ are open intervals} \\ \text{such that } A \subseteq \bigcup I_n \end{array} \right\}.$$

The question is how λ^*_δ depends on δ and whether $\lambda^*_\delta = \lambda^*$ for any value of $\delta > 0$.

__Lemma__ $\lambda^*_\delta(A) = \lambda^*(A)$ for all $\delta > 0$.

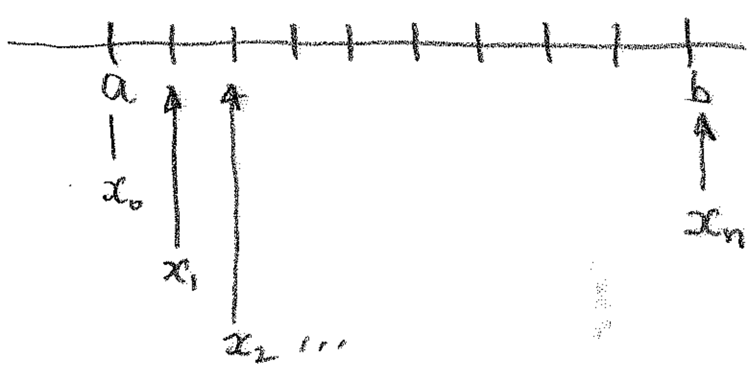
This is proved in the book on page 111 as Lemma 3.9. The proof depends on Lemma 3.8. Since the proof of 3.9 is in the text but the proof of 3.8 is not, I'll prove 3.8 now

__Lemma__ If $I = (a,b)$ is an open interval, $\delta > 0$ and $\varepsilon > 0$, Then there are a finite number of open intervals $I_1, I_2, \dots, I_n$ such that $I \subseteq \bigcup I_k$, $l(I_k) < \delta$ and $\sum l(I_n) < l(I) + \varepsilon$.

__Proof__, Let $n \in \mathbb{N}$ be chosen so large that

$$w = \frac{b-a}{n} < \delta .$$

Then define $x_k = a + wk = a + k\frac{b-a}{n}$.



Choose $\varepsilon_1 = \boxed{?} > 0$ and define $I_k = (x_k - \frac{\varepsilon_1}{2}, x_{k+1} + \frac{\varepsilon_1}{2})$.
Therefore $I \subseteq \bigcup_{k=1}^{n} I_k$. Moreover...

We will now write down some inequalities that allow us to solve for ε_1. Since

$$\ell(I_k) = x_{k+1} + \frac{\varepsilon_1}{2} - \left(x_k + \frac{\varepsilon_1}{2}\right) = w + \varepsilon_1 < \delta$$

then we want

$$0 < \varepsilon_1 < \delta - w.$$

Note that $w < \delta$ so $\delta - w > 0$. This means there are, in fact, values for $\varepsilon_1 > 0$ that satisfy the above inequality. also we want

$$\sum_{k=1}^{n} \ell(I_k) = n(w + \varepsilon_1) = b - a + n\varepsilon_1 < \ell(I) + \varepsilon$$

So

$$0 < \varepsilon_1 < \frac{\varepsilon}{n}.$$

Therefore we can take

$$\varepsilon_1 = \frac{1}{2} \min\left(\delta - w, \frac{\varepsilon}{n}\right)$$

and the proof of the lemma then follows