Math 713 Summary November 10, '10

There are two things left to cover before we begin integration:

① The Cantor function is continuous

② That $\mathcal{B} \subseteq \mathcal{M}$.

We have already shown that the cantor function ψ is monotone non-decreasing and onto $[0,1]$. To show it is continuous we first prove the more general result:

Lemma: If $f: \mathbb{R} \to \mathbb{R}$ is monotone and onto then it is continuous.

Proof: We assume f is non-decreasince since otherwise one could work with $-f$ since $-f$ is continuous if and only if f is continuous.

Let $x \in \mathbb{R}$ and define

$$L_x = \{f(p) : p < x\} \quad \text{and} \quad R_x = \{f(q) : q > x\}$$

since f in non-decreasing, then

$$\lim_{p \to x^-} f(p) = \sup L_x \quad \text{and} \quad \lim_{q \to x^+} f(q) = \inf R_x$$

Therefore, to show f is continuous at x it is enough to show that $\sup L_x = \inf R_x = f(x)$

Since f is monotone non-decreasing then

$$p < x < q \text{ implies } f(p) \leq f(x) \leq f(q)$$

It follows that

$$\sup\{f(p) : p < x\} \leq f(x) \leq \inf\{f(q) : q > x\}$$

or, in other words, that

$$\sup L_x \leq f(x) \leq \inf R_x$$

Suppose, for contradiction, that $\sup L_x < \inf R_x$. Then either

$$\sup L_x < f(x) \quad \text{or} \quad f(x) < \inf R_x$$

the details for $f(x) < \inf R_x$ are treated in the proof in the text, therefore we present the details for the case that $\sup L_x < f(x)$ here.

Case $\sup L_x < f(x)$: Let $y_0 \in \mathbb{R}$ be chosen so

$$\sup L_x < y_0 < f(x)$$

Since f is onto $\mathbb{R}$ there is $x_0 \in \mathbb{R}$ so $f(x_0) = y_0$.

Claim that $x_0 < x$. If not then $x \leq x_0$ would imply $f(x) \leq f(x_0)$ contradicting $f(x_0) = y_0 < f(x)$. Therefore $x_0 < x$. It follow that $f(x_0) \in L_x$. But then $f(x_0) \leq \sup L_x < f(x_0)$ is a contradiction. This implies the case $\sup L_x < f(x)$ can't happen.

Similarly, as shown in the text, the case $f(x) < \inf R_x$ can't happen. Therefore $\sup L_x = \inf R_x = f(x)$.

To prove that the Cantor function is continuous recall we have already shown that

The cantor function ψ maps $[0,1]$ onto $[0,1]$ and is non-decreasing.

Define

$$g(x) = \begin{cases} \psi(x) & \text{for } x \in [0,1] \\ x & \text{otherwise} \end{cases}$$

Then it is easy to check that g is non-decreasing and maps $\mathbb{R}$ onto $\mathbb{R}$.

Therefore g is continuous.

Since g is continuous then any restriction of g is also continuous. Thus

$$\psi = g|_{[0,1]}$$

the restriction of g to the interval $[0,1]$ is a continuous function.

We now turn to showing $\mathcal{B} \subseteq \mathcal{M}$.

In order to do this, it is enough to show $\tau \subseteq \mathcal{M}$, since $\mathcal{M}$ is a σ-algebra and $\mathcal{B}$ is the smallest σ-algebra that contains τ.

Thus we want to show for $O \in \tau$ that

$$\lambda^*(W) = \lambda^*(W \cap O) + \lambda^*(W \cap O^c) \quad \text{for every } W \subseteq \mathbb{R}.$$

We start with a lemma:

<u>Lemma</u>: If $A, B \subseteq \mathbb{R}$ and $d(A,B) > 0$ then $\lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B)$.

This lemma was stated last time but not proved. Recall that

$$d(A,B) = \inf\{|a-b| : a \in A \text{ and } b \in B\}$$

and

$$\lambda^*(A) = \lambda^*_\delta(A) = \inf\left\{\sum \ell(I_n) : \begin{array}{l} I_n \text{ are open intervals} \\ \text{with } \ell(I_n) < \delta \text{ and } A \subseteq \bigcup I_n \end{array}\right\}$$

for every $\delta > 0$.

After the break we will prove this lemma.

Proof: Let $\varepsilon > 0$ and $\delta = \text{dist}(A,B)$. Then there exists open intervals I_n such that $\ell(I_n) < \delta$, $A \cup B \subseteq \bigcup I_n$ and $\sum \ell(I_n) < \lambda^*(A \cup B) + \varepsilon$.

Define $\mathcal{J} = \{n : I_n \cap A \neq \emptyset\}$.

Then $A \subseteq \bigcup_{n \in \mathcal{J}} I_n$

Claim that $I_n \cap B = \emptyset$ for every $n \in \mathcal{J}$.

Suppose not. Then there would be some $n_0 \in \mathcal{J}$ such that $I_{n_0} \cap B \neq \emptyset$. In particular then there would be $a_0 \in I_{n_0} \cap A$ and $b_0 \in I_{n_0} \cap B$. But then

$$|a_0 - b_0| \leq \ell(I_n) < \delta = \text{dist}(A,B)$$

would contradict $\text{dist}(A,B)$ being a lower bound of the set $\{|a-b| : a \in A \text{ and } b \in B\}$.

Therefore $I_n \cap B = \emptyset$ for every $n \in \mathcal{J}$.

It follows that $B \subseteq \bigcup_{n \notin \mathcal{J}} I_n$.

Now

$$\lambda^*(A) + \lambda^*(B) \leq \sum_{n \in \mathcal{J}} \ell(I_n) + \sum_{n \notin \mathcal{J}} \ell(I_n) = \sum_{n \in \mathcal{J}} \ell(I_n)$$

$$< \lambda^*(A \cup B) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary then

$$\lambda^*(A) + \lambda^*(B) \leq \lambda^*(A \cup B).$$

Subadditivity implies

$$\lambda^*(A \cup B) \leq \lambda^*(A) + \lambda^*(B).$$

Therefore $\lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B)$.

In order to use this lemma to show $\tau \subseteq \mathcal{M}$ we will approximate $O \in \tau$ by a sequence of sets O_n such that the distance of O_n to the boundary of O is positive.

__Lemma__: Let $O \in \tau$ and define $O_n = \{x : d(x, O^c) > \frac{1}{n}\}$. Then

(a) O_n is open and $O_n \subseteq O$ for all $n \in \mathbb{N}$

(b) $O_1 \subseteq O_2 \subseteq O_3 \subseteq \cdots$ and $\bigcup_{n=1}^{\infty} O_n = O$

(c) If $O_n \neq \emptyset$ then $d(O_n, O^c) = 1/n$

(d) If $O_n \neq O$ then $d(O_n, O_{n+1}^c) = \frac{1}{n(n+1)}$

Note that the distance function $d(A,B)$ defined earlier is a function $d : \mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R}) \to \mathbb{R}^+$ where as the function d used in the definition of O_n has domain $\mathbb{R} \times \mathcal{P}(\mathbb{R})$.

The definition of $d : \mathbb{R} \times \mathcal{P}(\mathbb{R}) \to \mathbb{R}^+$ is

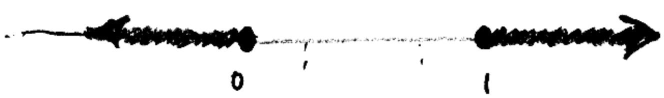
$$d(x, A) = d(\{x\}, A)$$

and using d to stand for two different but related functions should not cause any confusion.

Let's illustrate the theorem with an example

O 

we take $O = (0,1)$. Then

O^c 

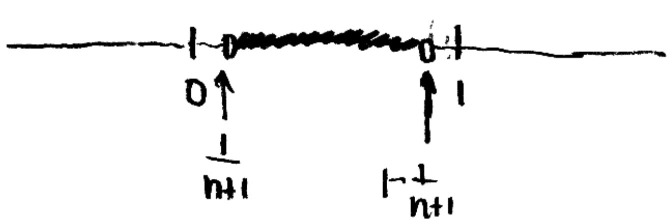
So

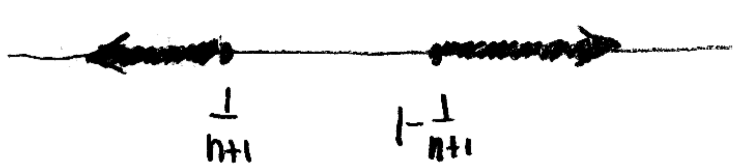
O_n 

Clearly $O_n = (\frac{1}{n}, 1-\frac{1}{n}) \subseteq O$ and $\bigcup_{n=1}^{\infty} O_n = O$. also

$$d(O_n, O^c) = \frac{1}{n}$$

Now

O_{n+1} 

an O_{n+1}^c 

Therefore $O_n \subseteq O_{n+1}$ and $d(O_n, O_{n+1}^c) = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$.

This example pretty much shows why the lemma is true, but it certainly isn't a proof.

Before proving a theorem, it is often helpful to think up some examples to see why the theorem is true. In research, sometimes, a general result may be too difficult to prove. Then a number of examples for which a specific version of a result can be shown to hold true may be interesting or even may be publishable.

For the current lemma, since open sets are anyway disjoint unions of open intervals the example with one interval makes the general result quite plausible.

The proof is in the book for parts (c) and (d) of the lemma. I wrote the proof of (a) and (b) down when I was reading the text last week. Please read the proof on page 113 and try to fill in the missing details yourself.