Math 713 Review

November 12-13, '10

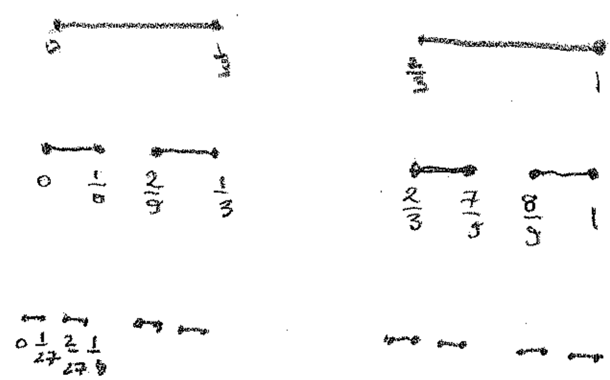
Write all the definitions on the review sheet:

1. The Cantor set

P_1

P_2

P_3



⋮

P_n is 2^n intervals of length $\frac{1}{3^n}$.

$$P = \bigcap_{n=1}^{\infty} P_n$$

Base 3 representation

$$P = \left\{ \sum_{k=1}^{\infty} \frac{a_k}{3^k} \; : \; a_k \in \{0, 2\} \right\}$$

2. The Lebesgue outer measure

$$\lambda^*(A) = \inf\left\{ \sum \ell(I_n) : \begin{array}{l} I_n \text{ are open intervals} \\ \text{such that } A \subseteq \bigcup I_n \end{array} \right\}$$

For $\delta > 0$

$$\lambda^*_\delta(A) = \inf\left\{ \sum \ell(I_n) : \begin{array}{l} I_n \text{ are open intervals with} \\ \ell(I_n) < \delta \text{ such that } A \subseteq \bigcup I_n \end{array} \right\}$$

Note $\lambda^*(A) = \lambda^*_\delta(A)$ for every $A \subseteq \mathbb{R}$ and $\delta > 0$.

3. Five definitions of Borel measurable functions

$\hat{C}$ is the smallest collection of functions that contains the continuous functions and is closed under pointwise limits.

$\mathcal{F}_1 = \{ f : \mathbb{R} \to \mathbb{R} \text{ st } f^{-1}(B) \in \mathcal{B} \text{ for } B \in \mathcal{B} \}$

$\mathcal{F}_2 = \{ f : \mathbb{R} \to \mathbb{R} \text{ st } f^{-1}(O) \in \mathcal{B} \text{ for } O \in \tau \}$

$\mathcal{F}_3 = \{ f : \mathbb{R} \to \mathbb{R} \text{ st } f^{-1}((-\infty, a)) \in \mathcal{B} \text{ for } a \in \mathbb{R} \}$

$\mathcal{F}_4 = \{ f : \mathbb{R} \to \mathbb{R} \text{ st } f^{-1}((a, \infty)) \in \mathcal{B} \text{ for } a \in \mathbb{R} \}$.

Note $\hat{C} = \mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \mathcal{F}_4$.

4. Two definitions of Borel Measurable sets

$\mathcal{B} = \{ B \subseteq \mathbb{R} : \chi_B \in \hat{C} \}$

$\mathcal{B}$ is the smallest σ-algebra containing $\mathcal{C}$.

5. Definition of Lebesgue measurable sets.

$\mathcal{M} = \{ E \subseteq \mathbb{R} : \lambda^*(W) = \lambda^*(W \cap E) + \lambda^*(W \cap E^c)$ for every $W \subseteq \mathbb{R} \}$.

6. Construction of S in Lemma 3.12.

An example of a non-measurable set is S defined as follows:

Let $x \sim y$ be the equivalence relation on $\mathbb{R}$ given by $x - y \in \mathbb{R}$.

$E_x = \{ y \in \mathbb{R} : y \sim x \}$

$\mathcal{C} = \{ E_x : x \in \mathbb{R} \}$

$w : \mathcal{C} \to \mathbb{R}$ such that $w(E_x) \in E_x$ where w is given by the axiom of choice.

$T = w(\mathcal{C})$.

$S = \{ z - [\![z]\!] : z \in T \}$ where $[\![z]\!]$ denotes the greatest integer less than or equal to z.

Say something about what it means for a collection of functions to be closed under pointwise limits.

Suppose $\mathcal{E}$ is a collection of functions which is closed under pointwise limits...

This means, for every sequence $f_n \in \mathcal{E}$ such that $f_n \to f$ pointwise to some f that $f \in \mathcal{E}$.

<u>Question</u>: If $\mathcal{E}$ is a finite collection then is it true or false that $\mathcal{E}$ is closed under pointwise limits?

First observation:

<u>Lemma</u>: Let $f_n \in \mathcal{E}$ be a pointwise convergent sequence and $x \in \mathbb{R}$. Then $f_n(x)$ is eventually constant.

<u>Proof</u>: Define $y_n = f_n(x)$. Then y_n is a Cauchy sequence. Define

$$A = \{ |y_i - y_j| : i,j \in \mathbb{N} \} \setminus \{0\}.$$

Since $\mathcal{E}$ is finite then $y_n = f_n(x)$ can take only finite many values. Therefore A is finite.

Since A is finite it has a minimum element

$$|y_{i_0} - y_{j_0}| = \min A.$$

Since $0 \notin A$ and $A \subseteq (0, \infty)$ then

$$\varepsilon = |y_{i_0} - y_{j_0}| > 0.$$

Since y_n is a Cauchy sequence there is $N \in \mathbb{N}$ large enough so that $p, q \geq N$ implies

$$|y_p - y_q| < \varepsilon.$$

However, if $y_p \neq y_q$ then

$$\varepsilon = \min A \leq |y_p - y_q| < \varepsilon$$

is a contradiction.

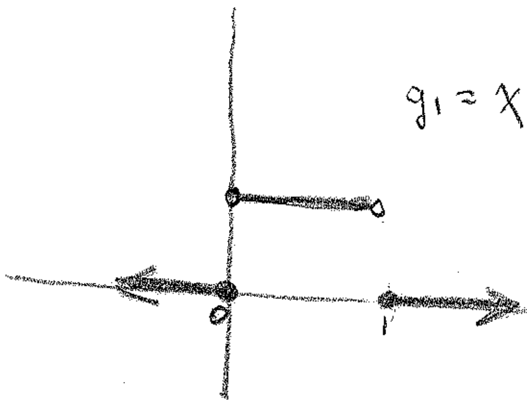
Therefore $y_p = y_q$ for $p, q \geq N$, or in otherwords the sequence y_n is eventually constant.

Can this lemma be extended to show that f_n is eventually a constant sequence of functions? Note that a constant sequence of functions is different than a sequence of constant functions!

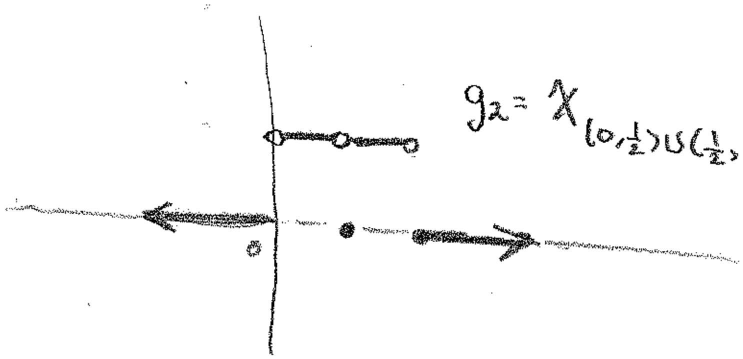
Lemma. Let $f_n \in \mathcal{E}$ such that f_n converges pointwise. Then f_n is eventually constant.

Example: Suppose $\mathcal{E}$ consisted of 3 functions

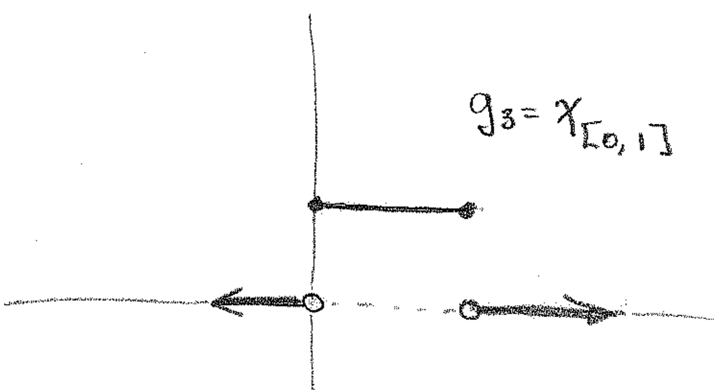
$g_1 = \chi_{(0,1)}$



$g_2 = \chi_{[0,\frac{1}{2})\cup(\frac{1}{2},0)}$



$g_3 = \chi_{[0,1]}$



In this example there is no single point $x_0 \in \mathbb{R}$ with the property that $g_i \neq g_j$ implies $g_i(x_0) \neq g_j(x_0)$.

Second observation

<u>lemma</u>: There exists a finite set $X \subseteq \mathbb{R}$ such that if $f_1, f_2 \in \mathcal{E}$ with $f_1 \neq f_2$ then $f_1(x) \neq f_2(x)$ for some $x \in X$.

<u>Proof</u>: Since $\mathcal{E}$ is finite it can be written as

$$\mathcal{E} = \{g_1, g_2, \dots, g_m\} \quad \text{for some } m \in \mathbb{N}$$

where each g_i denoted a distinct function.

Therefore for each $i \neq j$ we have $g_i \neq g_j$. This means there is some $x \in \mathbb{R}$ such that

$$g_i(x) \neq g_j(x).$$

For each $i \neq j$ let x_{ij} denote an x such that

$$g_i(x_{ij}) \neq g_j(x_{ij})$$

Then $X = \{x_{ij} : i \neq j\}$ is finite and satisfies the conditions given in the lemma.

It is now a matter of putting these facts together to prove that $\mathcal{E}$ is closed under pointwise limits. This problem is similar to the homework problem which stated that every finite collection of sets is a monotone class.

Being finite is a very strong hypothesis in analysis, as can be seen by these two results.

1. Every finite collection of sets is a monotone class
2. Every finite collection of functions is closed under pointwise limits.

An amazing amount of analytical structure comes automatically with the assumption of finiteness.

One question that can be asked...

> Are the assumptions similar to finiteness but which aren't quite so stringent that also imply similar amounts of analytic structure automatically?

Three such concepts are

(1) Countability
(2) Compactness
(3) Measure zero

Each is suitable in certain contexts. Note that a set which is finite is both countable, compact and has Lebesgue measure zero.

By the first lemma, for each $x \in X$ there is $N_x \in \mathbb{N}$ large enough so $f_n(x)$ is constant for $n \geq N_x$. Thus given $x \in X$ we have

$$f_n(x) = f_{N_x}(x) \quad \text{for} \quad n \geq N_x.$$

Let $N = \max\{N_x : x \in X\}$. Since X is finite this maximum exists and is finite. Moreover

$$f_n(x) = f_N(x) \quad \text{for all } n \geq N \text{ and } x \in X.$$

<u>Claim</u> that this implies $f_n = f_N$ for all $n \geq N$.

Suppose not. Then there is $n \geq N$ so $f_n \neq f_N$. Since $f_n, f_N \in \mathcal{E}$ there is i and j such that $f_n = g_i$ and $f_N = g_j$. Since $g_i \neq g_j$ then $i \neq j$. Moreover by definition of X we have $x = x_{ij} \in X$ such that $g_i(x) \neq g_j(x)$. However this implies $f_n(x) \neq f_N(x)$ for some $n \geq N$ and $x \in X$ which is a contradiction. This proves the claim.

Since f_n is eventually constant then

$$\lim_{n \to \infty} f_n = f_N \in \mathcal{E}$$

and so $\mathcal{E}$ is closed under pointwise limits.