Math 713 Summary

Nov 15, 2010

Today we finish everything leading up to the definition and development of the Lebesgue integral.

The only thing left is to show $\mathcal{B} \subseteq \mathcal{M}$.

Recall $\mathcal{M} = \{E \in \mathbb{R} : \lambda^*(W) = \lambda^*(W \cap E) + \lambda^*(W \cap E^c)$ for all $W \subseteq \mathbb{R}\}$

and $\mathcal{B}$ is the smallest σ-algebra that contains τ, the set of open sets. Since we have already shown that $\mathcal{M}$ is a σ-algebra it is enough to show that $\tau \subseteq \mathcal{M}$. For is $\tau \subseteq \mathcal{M}$ and $\mathcal{B}$ is the smallest σ-algebra which contains τ then it immediately follows that $\mathcal{B} \subseteq \mathcal{M}$.

To show $\tau \subseteq \mathcal{M}$ we need to show that each $O \in \tau$ satisfies the Carətheodory condition

$$\lambda^*(W) = \lambda^*(W \cap O) + \lambda^*(W \cap O^c) \quad \text{for all } W \subseteq \mathbb{R}.$$

This follows from the lemma

Lemma: If $A, B \subseteq \mathbb{R}$ and there is $O \in \tau$ such that $A \subseteq O$ and $B \subseteq O^c$ then $\lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B)$.

Assuming we can prove this lemma, then we can see that every $O \in \tau$ satisfies the Cartheodory condition as follows:

Let $O \in \mathcal{T}$ and $W \subseteq \mathbb{R}$.

Define $A = W \cap O$ and $B = W \cap O^c$.

Then $A \subseteq O$ and $B \subseteq O^c$, so by the lemma

$$\lambda^*(W) = \lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B)$$
$$= \lambda^*(W \cap O) + \lambda^*(W \cap O^c)$$

It follows that $O \in \mathcal{M}$.

Therefore $\mathcal{T} \subseteq \mathcal{M}$ and consequently $\mathcal{B} \subseteq \mathcal{M}$.

In order to prove the lemma we first prove

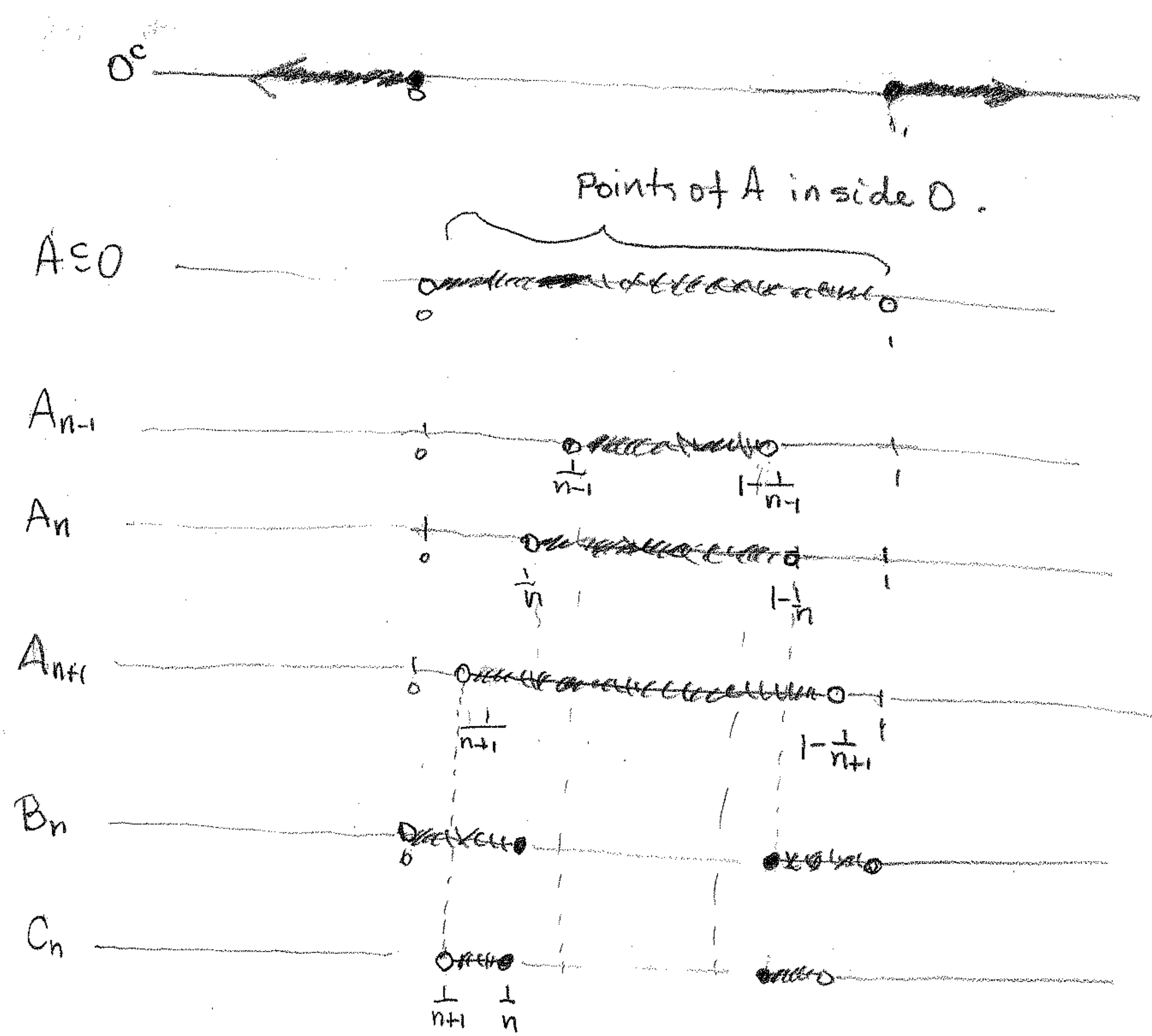
<u>Claim</u>: If $A \subseteq O$ and $O_n = \{x : d(x, O^c) > \frac{1}{n}\}$ then $\lambda^*(A) = \lim_{n \to \infty} \lambda^*(A \cap O_n)$.

For this proof we define the following sets

$$A_n = A \cap O_n$$
$$B_n = A \setminus A_n$$
$$C_n = A_{n+1} \setminus A_n$$

Let's draw an example of what these sets might look like, to help with our understanding of the proof.

Suppose $O=(0,1)$ and $A\subseteq O$. Then



Now $A_n \subseteq A_{n+1}$ implies $\lambda^*(A_n)\leq\lambda^*(A_{n+1})$ so $\lambda^*(A_n)$ is non-decreasing. Moreover $A_n\subseteq A$ implies $\lambda^*(A_n)\leq\lambda^*(A)$ so $\lambda^*(A_n)$ is bounded. A bounded monotone sequence converges therefore

$$\lim_{n\to\infty}\lambda^*(A_n)=\alpha\leq\lambda^*(A)$$

To finish the proof we need show $\lambda^*(A)\leq\alpha$.

Note that

$$A = A_n \cup B_n, \quad A_{n+1} = A_n \sqcup C_n \text{ and } B_n = \bigcup_{k=n}^{\infty} C_k .$$

Therefore

$$\lambda^*(A) \leq \lambda^*(A_n) + \lambda^*(B_n)$$
$$\leq \lambda^*(A_n) + \sum_{k=n}^{\infty} \lambda^*(C_k)$$

We now use the fact that

$$A_{n+1} = A_n \cup C_n \supseteq A_{n-1} \cup C_n$$

the observation that

$$d(A_{n-1}, C_n) = \frac{1}{n-1} - \frac{1}{n} = \frac{1}{n(n-1)} > 0$$

and the lemma from last week to obtain

$$\lambda^*(A_{n+1}) \geq \lambda^*(A_{n-1} \sqcup C_n) = \lambda^*(A_{n-1}) + \lambda^*(C_n)$$

Solving for $\lambda^*(C_n)$ we obtain

$$\lambda^*(C_n) \leq \lambda^*(A_{n+1}) - \lambda^*(A_{n-1})$$

Therefore

$$\sum_{k=n}^{m} \lambda^*(C_k) \leq \sum_{k=n}^{m} \left(\lambda^*(A_{k+1}) - \lambda^*(A_{k-1})\right).$$

During the break please this telescoping sum. Note that the sum telescopes with a difference of 2 so there should be 4 terms left after all the cancelations.

$$\sum_{k=n}^{m} \left(\lambda^*(A_{k+1}) - \lambda^*(A_{k-1})\right)$$

$$= \lambda^*(A_{m+1}) + \lambda^*(A_m) - \lambda^*(A_{n-1}) - \cancel{\lambda^*(A_n)}$$

Substituting this result we obtain.

$$\lambda^*(A) \leq \cancel{\lambda^*(A_n)} + \lim_{m\to\infty} \sum_{k=n}^{m} \lambda^*(C_k)$$

$$\leq \lim_{m\to\infty} \left(\lambda^*(A_{m+1}) + \lambda^*(A_m)\right) - \lambda^*(A_{n-1}).$$

$$= 2\alpha - \lambda^*(A_{n-1}).$$

Thus

$$\lambda^*(A) \leq 2\alpha - \lambda^*(A_{n-1})$$

Taking limits as $n \to \infty$ we obtain.

$$\lambda^*(A) \leq 2\alpha - \alpha = \alpha$$

which shows $\lambda^*(A) = \alpha$ and finishes the proof.

We are now ready to prove

<u>Lemma</u>: If $A, B \subseteq \mathbb{R}$ and there is $O \in \tau$ such that $A \subseteq O$ and $B \subseteq O^c$ then $\lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B)$

<u>Proof</u>: Define A_n as in the previous lemma.
Then $B \subseteq O^c$ implies $d(A_n, B) \geq \frac{1}{n} > 0$
Therefore

$$\lambda^*(A \cup B) \geq \lambda^*(A_n \cup B) = \lambda^*(A_n) + \lambda^*(B)$$

for all $n \in \mathbb{N}$.

Taking limits as $n \to \infty$

$$\lambda^*(A \cup B) \geq \lim_{n \to \infty} \lambda^*(A_n) + \lambda^*(B)$$

$$= \lambda^*(A) + \lambda^*(B)$$

by the previous lemma.

Since subadditivity implies $\lambda^*(A \cup B) \leq \lambda^*(A) + \lambda^*(B)$ then we have that

$$\lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B)$$

as required.