Lemma to Fatou's lemma: Suppose $f_n \to f$ pointwise.
Define $g_n = \inf\{f_k : k \geq n\}$.

Claim $g_n \to f$ pointwise.

Let $x \in \mathbb{R}$ and $\varepsilon > 0$.
Since $f_k \to f$ pointwise there is $N \in \mathbb{N}$ such that $|f_k(x) - f(x)| < \varepsilon/2$ for all $k \geq N$.

Claim $|g_n(x) - f(x)| < \varepsilon$ for all $n \geq N$.

Let $n \geq N$. By hypothesis

$$|f_k(x) - f(x)| < \varepsilon/2 \text{ for all } k \geq n.$$

Therefore

$$-\varepsilon/2 < f_k(x) - f(x) < \varepsilon/2 \quad \text{for all } k \geq n$$

Therefore

$$-\varepsilon/2 \leq \inf\{f_k(x) - f(x) : k \geq n\} \leq \varepsilon/2$$

Therefore

$$-\varepsilon/2 \leq g_n(x) - f(x) \leq \varepsilon/2$$

Or in other words

$$|g_n(x) - f(x)| \leq \varepsilon/2 < \varepsilon.$$