Math 713 Midterm Version A

1. Let $E \subseteq \mathbf{R}$. The closure of E is

(A) $\overline{E} = \{\, x \in \mathbf{R} : \text{for every } y \in E \text{ there exists } \epsilon > 0 \text{ such that } 0 < |y - x| < \epsilon \,\}$

(B) $\overline{E} = \{\, x \in \mathbf{R} : \text{for every } \epsilon > 0 \text{ there exists } y \in E \text{ such that } 0 < |y - x| < \epsilon \,\}$

(C) $\overline{E} = \{\, x \in \mathbf{R} : \text{for every } y \in E \text{ there exists } \epsilon > 0 \text{ such that } |y - x| < \epsilon \,\}$

(D) $\overline{E} = \{\, x \in \mathbf{R} : \text{for every } \epsilon > 0 \text{ there exists } y \in E \text{ such that } |y - x| < \epsilon \,\}$

(E) none of these

2. Given an interval I let $\ell(I)$ be its length. For each subset $A \subseteq \mathbf{R}$, the Lebesgue outer measure of A, denoted by $\lambda^*(A)$, is defined by

(A) $\lambda^*(A) = \inf \left\{ \sum_n \ell(I_n) : I_n \text{ are open intervals where } A \subseteq \bigcap_n I_n \right\}$

(B) $\lambda^*(A) = \inf \left\{ \sum_n \ell(I_n) : I_n \text{ are open intervals where } A \subseteq \bigcup_n I_n \right\}$

(C) $\lambda^*(A) = \sup \left\{ \sum_n \ell(I_n) : I_n \text{ are open intervals where } A \supseteq \bigcap_n I_n \right\}$

(D) $\lambda^*(A) = \sup \left\{ \sum_n \ell(I_n) : I_n \text{ are open intervals where } A \supseteq \bigcup_n I_n \right\}$

(E) none of these

3. Let $\hat{C}$ be the collection of Borel measurable functions, $\mathcal{B}$ the set of Borel measurable sets and τ the collection of open sets in $\mathbf{R}$. Then

(A) $\hat{C} = \{\, f : \mathbf{R} \to \mathbf{R} \text{ such that } f^{-1}(B) \in \mathcal{B} \text{ for every } B \in \mathcal{B} \,\}$

(B) $\hat{C} = \{\, f : \mathbf{R} \to \mathbf{R} \text{ such that } f^{-1}(O) \in \mathcal{B} \text{ for every } O \in \tau \,\}$

(C) $\hat{C} = \{\, f : \mathbf{R} \to \mathbf{R} \text{ such that } f^{-1}(a) \in \mathcal{B} \text{ for every } a \in \mathbf{R} \,\}$

(D) both (A) and (B)

(E) both (A), (B) and (C)

4. Let $\mathcal{A}$ be the smallest σ-algebra that contains the open sets, $\hat{C}$ the collection of Borel measurable functions and λ^* the Lebesgue outer measure. Then

(A) $\mathcal{A} = \{\, B \subseteq \mathbf{R} : \chi_B \in \hat{C} \,\}$

(B) $\mathcal{A} = \{\, E \subseteq \mathbf{R} : \lambda^*(W) = \lambda^*(W \cap E) + \lambda^*(W \cap E^c) \text{ for every } W \subseteq \mathbf{R} \,\}$

(C) $\mathcal{A} = \{\, W \subseteq \mathbf{R} : \lambda^*(W) = \lambda^*(W \cap E) + \lambda^*(W \cap E^c) \text{ for every } E \subseteq \mathbf{R} \,\}$

(D) both (A) and (B)

(E) both (A), (B) and (C)

5. Let P be the Cantor set. Then

(A) $P = \bigcap_{n=1}^{\infty} P_n$ where $P_n = \Big\{ \sum_{k=1}^{n} \frac{a_k}{3^k} : a_k \in \{0,1,2\} \Big\}$

(B) $P = \bigcap_{n=1}^{\infty} P_n$ where $P_n = \Big\{ \sum_{k=1}^{n} \frac{a_k}{3^k} : a_k \in \{0,2\} \Big\}$

(C) $P = \Big\{ \sum_{k=1}^{\infty} \frac{a_k}{3^k} : a_k \in \{0,2\} \Big\}$

(D) $P = \Big\{ \sum_{k=1}^{\infty} \frac{a_k}{2^k} : a_k \in \{0,1\} \Big\}$

(E) none of these

6. In the following true false questions $\hat{C}$ is the set of Borel measurable functions, $\mathcal{B}$ is the collection of Borel measurable sets, λ^* is the Lebesgue outer measure and $\mathcal{M}$ is the set of Lebesgue measurable sets.

(i) Let $A, B \in \mathcal{B}$ such that $A \subseteq B$ and $A \neq B$. Then $\lambda^*(A) < \lambda^*(B)$.

(A) true

(B) false

(ii) If $f : \mathbf{R} \to \mathbf{R}$ is a continuous function then $f \in \hat{C}$.

(A) true

(B) false

(iii) Suppose

$$x = \sum_{k=1}^{\infty} \frac{d_k}{10^k} \quad \text{and} \quad y = \sum_{k=1}^{\infty} \frac{e_k}{10^k} \quad \text{where} \quad d_k, e_k \in \{0,1,2,\ldots,9\}.$$

If $x < y$ then $d_1 \leq e_1$.

(A) true

(B) false

(iv) If $\lambda^*(E) = 0$ and $E \in \mathcal{M}$ then E must be countable.

(A) true

(B) false

7. Give an example of a set which is not Lebesgue measurable.

8. Let $A, B \subseteq \mathbf{R}$. Prove or find a counter example to the claim that $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

9. [Extra Credit] We have studied the mathematics of Borel, Cantor, Lebesgue and Riemann. Order these mathematicians according to when they were born.

10. Let $d(A,B) = \inf\{\, |a-b| : a \in A \text{ and } b \in B \,\}$ and $\mathcal{M}$ be the collection of Lebesgue measurable sets. Prove one of the following:

(i) $\lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B)$ for $A, B \subseteq \mathbf{R}$ with $d(A,B) > 0$.

(ii) $\lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B)$ for $A, B \in \mathcal{M}$ with $A \cap B = \emptyset$.

11. Prove one of the following:

(i) Let f be monotone function which maps $\mathbf{R}$ onto $\mathbf{R}$. Then f is continuous.

(ii) Let f_n be a sequence of continuous functions which map $\mathbf{R}$ into $\mathbf{R}$. Suppose $f_n \to f$ uniformly. Then f is continuous.

12. Let $d(A,B) = \inf\{\, |a-b| : a \in A \text{ and } b \in B\,\}$ and $\mathcal{M}$ be the collection of Lebesgue measurable sets. Prove or find a counter example to one of the following claims:

(i) Let $\mathcal{C}$ be a finite collection of non-empty subsets of $\mathbf{R}$. Given $A, B \in \mathcal{C}$ suppose $A \simeq B$ means that $d(A,B) = 0$. Prove or find a counter example to the claim that $\simeq$ is an equivalence relation on $\mathcal{C}$.

(ii) Suppose $A, B \in \mathcal{M}$ are such that $A \subseteq B$ and $\lambda^*(A) < \infty$. Prove or find a counter example to the claim that $\lambda^*(B \setminus A) = \lambda^*(B) - \lambda^*(A)$.