Math 713 Quiz 1 Version B

1. Let $E = (1,2] \cup \{5\} \cup [7,9)$. Then

(A) $\overline{E} = \emptyset$

(B) $\overline{E} = [1,9]$

(C) $\overline{E} = [1,2] \cup [7,9]$

(D) $\overline{E} = [1,2] \cup \{5\} \cup [7,9]$

(E) none of these

2. Let $D \subseteq \mathbf{R}$. A set $U \subseteq D$ is open relative to D if and only if

(A) for every sequence $x_n \in U^c$ and $x \in \mathbf{R}$ then $x_n \to x$ implies $x \in U^c$

(B) for every $x \in U$ there exists $r > 0$ such that $(x-r, x+r) \subseteq U$

(C) for every $x \in U$ there exists $r > 0$ such that $(x-r, x+r) \cap D \subseteq U$

(D) for every $x \in U$ there exists $r > 0$ such that $(x-r, x+r) \subseteq U \cap D$

(E) none of these

3. Let $J_n = (0, 1/n)$ for $n \in \mathbf{N}$ and $V = \bigcap_{n=1}^{\infty} J_n$. Then

(A) $V = \emptyset$

(B) $V = \{0\}$

(C) $V = [0,1)$

(D) $V = [0,1]$

(E) none of these

4. Let $f\colon X \to Y$. If $A \subseteq Y$ then

(A) $f^{-1}(A) = \{ f(x) : x \in X \}$

(B) $f^{-1}(A) = \{ x \in X : f(x) \in Y \}$

(C) $f^{-1}(A) = \{ x \in X : f(x) \in A \}$

(D) $f^{-1}(A) = \{ x \in X : f(x) \in Y \setminus A \}$

(E) none of these

5. [Extra Credit] State the first and last names of three world famous mathematicians or statisticians either dead or alive who do not work at UNR.

Sir Ronald Fisher, Karl Pearson, Carl Gauss

6. Let $E \subseteq \mathbf{R}$. A real number x is called an accumulation point of E if for each $\epsilon > 0$ there is $y \in E$ such that $0 < |y - x| < \epsilon$. Let

$$E' = \{\, x \in \mathbf{R} : x \text{ is an accumulation point of } E \,\}.$$

If $E = (1, 2] \cup \{\, 5 \,\} \cup [7, 9)$, then

(A) $E' = \emptyset$

(B) $E' = [1, 9]$

(C) $E' = [1, 2] \cup [7, 9]$

(D) $E' = [1, 2] \cup \{\, 5 \,\} \cup [7, 9]$

(E) none of these

7. Let Ω be a set and $\mathcal{A}$ be a nonempty collection of subsets of Ω such that

(i) $D_n \in \mathcal{A}$ and $D_1 \subseteq D_2 \subseteq \cdots$ implies $\bigcap_{n=1}^{\infty} D_n \in \mathcal{A}$

(ii) $D_n \in \mathcal{A}$ and $D_1 \supseteq D_2 \supseteq \cdots$ implies $\bigcup_{n=1}^{\infty} D_n \in \mathcal{A}$

then $\mathcal{A}$ must be

(A) a σ-algebra

(B) a monotone class

(C) both (A) and (B)

(D) none of these

8. A sequence of real numbers converges if and only if

(A) it is a Cauchy sequence

(B) every subsequence converges

(C) it has exactly one cluster point

(D) both (A) and (B)

(E) both (A), (B) and (C)

9. Let $D \subseteq \mathbf{R}$ and $f_n : D \to \mathbf{R}$ for $n \in \mathbf{N}$. Suppose for each $x \in D$ and $\epsilon > 0$ there is $N \in \mathbf{N}$ such that $n, m \geq N$ implies $|f_n(x) - f_m(x)| < \epsilon$. Then the sequence f_n of real valued functions must be

(A) pointwise convergent

(B) uniformly convergent

(C) differentiable

(D) both (A) and (B)

(E) both (A), (B) and (C)

10. Fill in the missing blank in the statement of the following proposition.

Proposition 2.14. A set is $\boxed{\text{open}}$ if and only if its complement is closed.

11. Fill in the missing blanks in the statement of the following theorems.

Monotone Class Theorem Let Ω be a set and $\mathcal{A}_0$ an algebra of subsets of Ω. Let $\mathcal{D}$ be a collection of subsets of Ω such that $\mathcal{D} \supseteq \mathcal{A}_0$ and $\mathcal{D}$ is a monotone class. Then $D \supseteq$ $\boxed{\mathcal{A}(\mathcal{A}_0) \text{ the } \sigma\text{-algebra generated by } \mathcal{A}_0}$.

Theorem 2.7. A $\boxed{\text{bounded}}$ function on $[a, b]$ is Riemann integrable if and only if the set of points of discontinuity of the function has measure $\boxed{\text{zero}}$.

12. Suppose A and B are open subsets of $\mathbf{R}$ such that $A \cap \mathbf{Q} = B \cap \mathbf{Q}$. Prove or disprove the claim that $A = B$.

False $A = (0, \sqrt{2}) \cup (\sqrt{2}, 4)$
and $B = (0, 4)$

Then $A \cap \mathbb{Q} = B \cap \mathbb{Q}$ and A and B are both open but $A \neq B$.

13. Let $D \subseteq \mathbf{R}$ and $f_n \colon D \to \mathbf{R}$ be a sequence of continuous functions. Suppose $f_n \to f$ uniformly. Prove that f is continuous.

Let $x_0 \in D$.

Given $\varepsilon > 0$. Since $f_n \to f$ uniformly there is $N \in \mathbb{N}$ such that $n \geq N$ and $x \in D$ implies $|f_n(x) - f(x)| < \varepsilon/3$.

Since f_N is continuous there is $\delta > 0$ such that $x \in D$ and $|x - x_0| < \delta$ implies $|f_N(x) - f_N(x_0)| < \varepsilon/3$.

Thus for $x \in D$ and $|x - x_0| < \delta$ we have

$$|f(x) - f(x_0)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)|$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Therefore f is continuous at x_0 for any $x_0 \in D$.

This shows that f is continuous.

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14. Prove one of the following results.

(i) Let $U \subseteq \mathbf{R}$ be a non-empty bounded open set. Given a real number $x \in U$ define $a = \inf\{\, y : y < x \text{ and } (y, x) \subseteq U \,\}$. Show that $a \notin U$.

(ii) Let $\mathcal{A}_0$ be an algebra and $\mathcal{F}$ the smallest monotone class that contains $\mathcal{A}_0$. Given $A \in \mathcal{A}_0$ define $\mathcal{E} = \{\, F \in \mathcal{F} : A \cup F \in \mathcal{F} \,\}$. Show that $\mathcal{E} = \mathcal{F}$.

Proof of (i) same as problem 13 on the practice quiz.
Proof of (ii):
Claim $\mathcal{A}_0 \subseteq \mathcal{E}$. Let $B \in \mathcal{A}_0$. Then $B \cup A \in \mathcal{A}_0$ since $\mathcal{A}_0$ is an algebra. Since $\mathcal{A}_0 \subseteq \mathcal{F}$ we have $B \in \mathcal{F}$ and $B \cup A \in \mathcal{F}$ therefore $B \in \mathcal{E}$ by definition. Thus $\mathcal{A}_0 \subseteq \mathcal{E}$.

Claim $\mathcal{E}$ is a monotone class. Suppose $F_i \in \mathcal{E}$ and that $F_1 \subseteq F_2 \subseteq \cdots$. Since $\mathcal{E} \subseteq \mathcal{F}$ and $\mathcal{F}$ is a monotone class then $\bigcup_{i=1}^{\infty} F_i \in \mathcal{F}$. By definition $F_i \in \mathcal{E}$ implies $A \cup F_i \in \mathcal{F}$. Since $A \cup F_1 \subseteq A \cup F_2 \subseteq \cdots$ and $\mathcal{F}$ is a monotone class then $\bigcup_{i=1}^{\infty} (A \cup F_i) \in \mathcal{F}$.
Now $\bigcup_{i=1}^{\infty} F_i \in \mathcal{F}$ and $A \cup \bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} (A \cup F_i) \in \mathcal{F}$ implies $\bigcup_{i=1}^{\infty} F_i \in \mathcal{E}$.

Similarly if $F_1 \supseteq F_2 \supseteq \cdots$ then $\bigcap_{i=1}^{\infty} F_i \in \mathcal{F}$. Again $A \cup F_i \in \mathcal{F}$ so that $A \cup F_1 \supseteq A \cup F_2 \supseteq \cdots$ imply $\bigcap_{i=1}^{\infty} (A \cup F_i) \in \mathcal{F}$.
Now $\bigcap_{i=1}^{\infty} F_i \in \mathcal{F}$ and $A \cup \bigcap_{i=1}^{\infty} F_i = \bigcap_{i=1}^{\infty} (A \cup F_i) \in \mathcal{F}$ implies $\bigcap_{i=1}^{\infty} F_i \in \mathcal{E}$.

It follow that $\mathcal{E}$ is a monotone class containing $\mathcal{A}_0$. Since $\mathcal{F}$ is the smallest monotone class containing $\mathcal{A}_0$ we have $\mathcal{F} \subseteq \mathcal{E}$. Then $\mathcal{E} \subseteq \mathcal{F}$ implies $\mathcal{E} = \mathcal{F}$.

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15. Prove or disprove one of the following claims.

(i) Let $f: \mathbf{R} \to \mathbf{R}$. If $f^{-1}(F)$ is closed for every closed $F \subset \mathbf{R}$ prove or disprove the claim that f is continuous.

(ii) Let $E \subseteq \mathbf{R}$ be nonempty and

$$F = \{ x \in \overline{E} : \text{there exists } \epsilon > 0 \text{ such that } (x, x+\epsilon) \cap E = \emptyset \}$$

Prove or disprove the claim that F is countable.

<u>Proof of (i)</u>: Let $O \subseteq \mathbb{R}$ be open, Claim that $f^{-1}(O)$ is also open. Let $F = O^c$. Since the complement of an open set is closed then F is closed. By hypothesis $f^{-1}(F)$ is closed. Thus

$$f^{-1}(O) = f^{-1}(F^c) = (f^{-1}(F))^c$$

is open because $(f^{-1}(F))^c$ being the complement of the closed set $f^{-1}(F)$ is open. Thus f is continuous.

Proof of (ii):

For each $x \in F$ let $\varepsilon_x > 0$ be chosen so $I_x = (x, x+\varepsilon_x) \cap E = \emptyset$.
Define $\mathcal{C} = \{I_x : x \in F\}$.

Claim. If $I_x \cap I_y \neq \emptyset$ then $x = y$.

If not, then for some $x < y$ we have $I_x \cap I_y \neq \emptyset$.

If $I_x \cap I_y = \emptyset$ then either $x + \varepsilon_x \leq y$ or $y + \varepsilon_y \leq x$.
Thus $I_x \cap I_y \neq \emptyset$ implies both $x + \varepsilon_x > y$ and $y + \varepsilon_y > x$.
Therefore $x < y < x + \varepsilon_x$ implies $y \in I_x$. Since I_x is open there is $r > 0$ such that $(y-r, y+r) \subseteq I_x$.
Since $y \in \overline{E}$ there is $z \in E$ such that $|y - z| < r$. It follows that $z \in I_x$ which contradicts $I_x \cap E = \emptyset$.
Therefore $I_x \cap I_y \neq \emptyset$ implies $x = y$.

For each $I_x \in \mathcal{C}$ choose $r_x \in \mathbb{Q} \cap I_x$. By the claim we have that $r_x = r_y$ implies $x = y$.

Let $M = \{r_x : x \in F\}$. Then $x \to r_x$ is a one-to-one function from F onto M. Since $M \subseteq \mathbb{Q}$ then M is countable. It follows that F is countable.