Math 713 Practice Quiz 1 Version A

1. Let $I_n = [-n, 1/n]$ for $n \in \mathbf{N}$ and $U = \bigcup_{n=1}^{\infty} I_n$. Then
   - (A) $U = (-\infty, 0)$
   - (B) $U = (-\infty, 0]$
   - (C) $U = (-\infty, 1)$
   - (D) $U = (-\infty, 1]$
   - (E) none of these

2. Let $J_n = [0, 1/n]$ for $n \in \mathbf{N}$ and $V = \bigcap_{n=1}^{\infty} J_n$. Then
   - (A) $V = \emptyset$
   - (B) $V = \{0\}$
   - (C) $V = [0, 1)$
   - (D) $V = [0, 1]$
   - (E) none of these

3. Let $f: X \to Y$. If $A \subseteq X$ then
   - (A) $f(A) = \{ f(x) : x \in X \}$
   - (B) $f(A) = \{ x \in X : f(x) \in Y \}$
   - (C) $f(A) = \{ x \in X : f(x) \in A \}$
   - (D) $f(A) = \{ x \in X : f(x) \in Y \setminus A \}$
   - (E) none of these

4. Let Ω be a set and $\mathcal{A}$ be a collection of subsets of Ω such that $A \in \mathcal{A}$ implies $A^c \in \mathcal{A}$ and $A, B \in \mathcal{A}$ implies $A \cup B \in \mathcal{A}$. Then $\mathcal{A}$ must be
   - (A) an algebra
   - (B) a σ-algebra
   - (C) a monotone class
   - (D) both (A) and (B)
   - (E) both (A), (B) and (C)

5. [Extra Credit] State the first and last names of three world famous mathematicians either dead or alive who do not work at UNR. Correct spelling is essential.

Georg Cantor, Bernhard Riemann
Émile Borel

6. A set $U \in \mathbf{R}$ is open if for every $x \in U$ there exists $r > 0$ such that $(x-r, x+r) \subseteq U$. This is equivalent to saying U is open if

   (A) for every sequence $x_n \in U$ and $x \in \mathbf{R}$ then $x_n \to x$ implies $x \in U$

   (B) for every sequence $x_n \in U^c$ and $x \in \mathbf{R}$ then $x_n \to x$ implies $x \in U^c$

   (C) for every $x \in U^c$ there exists $r > 0$ such that $U^c \subseteq [x-r, x+r]$

   (D) every sequence $x_n \in U$ has a convergent subsequence

   (E) none of these

7. Let $D \subseteq \mathbf{R}$ and $f: D \to \mathbf{R}$. Suppose for every $\epsilon > 0$ there is $\delta > 0$ such that $a, b \in D$ and $|a - b| < \delta$ implies $|f(a) - f(b)| < \epsilon$. Then f must be

   (A) continuous

   (B) uniformly continuous

   (C) differentiable

   (D) both (A) and (B)

   (E) both (A), (B) and (C)

8. Let x_n be a sequence of real numbers. A real number x is said to be a cluster point of x_n if for each $\epsilon > 0$ and $N \in \mathbf{N}$ there is an $n \geq N$ such that $|x - x_n| < \epsilon$. This is equivalent to saying $x \in \mathbf{R}$ is a cluster point of x_n if

   (A) $x \in \overline{E}$ where $E = \{ x_n : n \in \mathbf{N} \}$

   (B) there exists a subsequence x_{n_k} such that $x_{n_k} \to x$

   (C) there exists a subsequence x_{n_k} of distinct elements such that $x_{n_k} \to x$

   (D) $x \in [\alpha, \beta]$ where $\alpha = \liminf x_n$ and $\beta = \limsup x_n$

   (E) none of these

9. Let $D \subseteq \mathbf{R}$ and $f_n: D \to \mathbf{R}$ for $n \in \mathbf{N}$. Suppose for each $x \in D$ and $\epsilon > 0$ there is $N \in \mathbf{N}$ such that $n, m \geq N$ implies $|f_n(x) - f_m(x)| < \epsilon$. Then the sequence f_n of real valued functions must be

   (A) pointwise convergent

   (B) uniformly convergent

   (C) differentiable

   (D) both (A) and (B)

   (E) both (A), (B) and (C)

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10. Fill in the missing blanks in the statement of the following axiom.

Completeness Axiom. A non-empty subset of real numbers that is bounded above has a least upper bound.

11. Fill in the missing blanks in the statement of the following theorem.

Theorem 2.7. A bounded function on $[a, b]$ is Riemann integrable if and only if the set of points of discontinuity of the function has measure zero.

12. Show there is an irrational number between any two rational numbers.

Let $r_1, r_2 \in \mathbb{Q}$ such that $r_1 < r_2$.

Define $x = \dfrac{r_1 + \sqrt{2}\, r_2}{1+\sqrt{2}}$

Then $\dfrac{r_1+\sqrt{2}\,r_2}{1+\sqrt{2}} < \dfrac{r_2+\sqrt{2}\,r_2}{1+\sqrt{2}} = r_2$ implies $x < r_2$

$\dfrac{r_1+\sqrt{2}\,r_2}{1+\sqrt{2}} > \dfrac{r_1+\sqrt{2}\,r_1}{1+\sqrt{2}} = r_1$ implies $x > r_2$

so $x \in (r_1, r_2)$. Claim x is irrational. Suppose not then $\dfrac{r_1+\sqrt{2}\,r_2}{1+\sqrt{2}} = \dfrac{p}{q}$ $\quad q r_1 + q\sqrt{2}\, r_2 = p + p\sqrt{2}$

so $\sqrt{2}(q r_2 - p) = q r_1 - p$. Since $\frac{p}{q} < r_2$ then $q r_2 - p \neq 0$

Thus $\sqrt{2} = \dfrac{q r_1 - p}{q r_2 - p}$ contradicting $\sqrt{2}$ is irrational.

13. Let $U \subseteq \mathbf{R}$ be a bounded open set and $x \in U$. Define

$$a = \inf\{\, y : y < x \text{ and } (y, x) \subseteq U \,\}.$$

Show that $a \notin U$.

Let $A = \{y : y < x \text{ and } (y,x) \subseteq U\}$

Since $x \in U$ and U is open there is $r_1 > 0$ such that $(x - r_1, x + r_1) \subseteq U$. Therefore $x - r_1 \in A$ and $a \leq x - r_1$.

For contradiction suppose $a \in U$. Thus there is $r_2 > 0$ such that $(a - r_2, a + r_2) \subseteq U$.

Let $r = \min(r_1, r_2)$. Thus $(a - r, a + r) \subseteq U$ and

$$a + r < a + r_1 \leq x - r_1 + r = x$$

Moreover, since $a + r > a$ and a is the greatest lower bound, then $a + r$ could not be a lower bound of A. Therefore there is $y < x$ such that $a \leq y < a + r$ and $(y, x) \subseteq U$.

Since $(a - r, a + r) \subseteq U$ and $(y, x) \subseteq U$ then

$$(a - r, a + r) \cup (y, x) \subseteq U.$$

Now $a - r < y < a + r < x$ implies $(a - r, x) \subseteq U$. Therefore $a - r \in A$. But this contradicts that a is a lower bound of A. Thus $a \notin U$.

14. Prove one of the following theorems.

(i) Let $D \subseteq \mathbf{R}$ and $f_n: D \to \mathbf{R}$ be a sequence of continuous functions. Suppose $f_n \to f$ uniformly. Then f is continuous.

(ii) Suppose $f: [0,1] \to \mathbf{R}$ is continuous and $f(c) > 0$ for some $c \in (0,1)$. Show there is $h > 0$ such that $|x - c| < h$ implies $f(x) > 0$.

Proof of (i). Let $x_0 \in D$ and $\varepsilon > 0$. Since $f_n \to f$ uniformly there is $N \in \mathbb{N}$ such that $n \geq N$ and $z \in D$ implies $|f_n(z) - f(z)| < \varepsilon/3$.

Since f_N is continuous there is $\delta > 0$ such that $x \in D$ and $|x - x_0| < \delta$ implies $|f_N(x) - f_N(x_0)| < \varepsilon/3$.

It follow for $x \in D$ and $|x - x_0| < \delta$ that

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - f_N(x) + f_N(x) - f_N(x_0) + f_N(x_0) - f(x_0)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

Proof of (ii) Let $\varepsilon = f(c)/2 > 0$. Since f is continuous at c there is $\delta > 0$ such that $x \in [0,1]$ and $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$.

Let $h = \min(\delta, c, 1-c)$. Then $|x - c| < h$ implies that $x \in [0,1]$ and $|x - c| < \delta$. Therefore

$$\begin{aligned} f(x) = f(c) + f(x) - f(c) &\geq f(c) - |f(x) - f(c)| \\ &> f(c) - \varepsilon = f(c)/2 > 0. \end{aligned}$$

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15. Prove or find a counter example to one of the following claims.

(i) Let $x_n \in \mathbf{R}$ for $n \in \mathbf{N}$ and $h: \mathbf{N} \to \mathbf{N}$ be a bijection. Define $y_n = x_{h(n)}$. Let $E = \{ x \in \mathbf{R} : x \text{ is a cluster point of } x_n \}$ and $F = \{ y \in \mathbf{R} : y \text{ is a cluster point of } y_n \}$. Prove or find a counter example to the claim that $E = F$.

(ii) For $A, B \subseteq \mathbf{R}$ define $A \cdot B = \{ ab : a \in A \text{ and } b \in B \}$. Prove or find a counter example to the claim that $\overline{A} \cdot \overline{B} = \overline{A \cdot B}$.

<u>Proof of (i)</u>: Let $y \in F$. Then there is a subsequence $y_{n_k} \to y$. Let $g: \mathbb{N} \to \mathbb{N}$ be defined by $g(k) = h(n_k)$. Since h is a bijection then g is injective. It follows that $g^{-1}(\{1, 2, \ldots, h(n_m)\})$ is finite for each m. Therefore

$$B_m = \{ k > m : h(n_k) > h(n_m) \}$$
$$= \mathbb{N} \setminus \left(g^{-1}(\{1, 2, \ldots, h(n_m)\}) \cup \{1, 2, \ldots, m\} \right)$$

is infinite and in particular non-empty for each m.

Let $k_1 = 1$ and $k_{j+1} = \min B_{k_j}$ for $j \in \mathbb{N}$. By definition of B_m we have $k_{j+1} > k_j$ and $h(n_{k_{j+1}}) > h(n_{k_j})$ for $j \in \mathbb{N}$. Therefore $x_{h(n_{k_j})}$ is a subsequence of x_n. Since

$$x_{h(n_{k_j})} = y_{n_{k_j}} \to y \quad \text{as } j \to \infty$$

it follows that $F \subseteq E$.

To see that $E \subseteq F$ note that $h^{-1}: \mathbb{N} \to \mathbb{N}$ is a bijection and so we also have $x_n = y_{h^{-1}(n)}$. Now following the same argument as above we obtain $F \subseteq E$ and so $E = F$.

Counterexample of (ii)

Let $A = \left\{ \frac{1}{2n+1} : n \in \mathbb{N} \right\}$ and $B = \mathbb{N}$

Then $\frac{n}{2n+1} \subseteq A \cdot B$ for each n.

Since $\frac{n}{2n+1} \to \frac{1}{2}$ it follows that $\frac{1}{2} \in \overline{A \cdot B}$.

However $\bar{A} = \{0\} \cup \left\{ \frac{1}{2n+1} : n \in \mathbb{N} \right\}$

and $\bar{B} = \mathbb{N}$.

Therefore $\frac{1}{2} \notin \bar{A} \cdot \bar{B}$.