Math 713 Practice Quiz 1 Version A

15. Prove or find a counter example to one of the following claims.

(i) Let $x_n \in \mathbf{R}$ for $n \in \mathbf{N}$ and $h: \mathbf{N} \to \mathbf{N}$ be a bijection. Define $y_n = x_{h(n)}$. Let $E = \{ x \in \mathbf{R} : x \text{ is a cluster point of } x_n \}$ and $F = \{ y \in \mathbf{R} : y \text{ is a cluster point of } y_n \}$. Prove or find a counter example to the claim that $E = F$.

(ii) For $A, B \subseteq \mathbf{R}$ define $A \cdot B = \{ ab : a \in A \text{ and } b \in B \}$. Prove or find a counter example to the claim that $\overline{A} \cdot \overline{B} = \overline{A \cdot B}$.

Proof of (i)

x is a cluster point of x_n if and only if for each $\varepsilon > 0$, infinitely many terms of the sequence are within ε of x.

Let $x \in E$.

Then $X_\varepsilon = \{ n : |x_n - x| < \varepsilon \}$ is infinite for every $\varepsilon > 0$

Claim $Y_\varepsilon = \{ n : |y_n - x| < \varepsilon \}$ is infinite for every $\varepsilon > 0$.

Since

$$\begin{aligned} h(Y_\varepsilon) &= \{ h(n) : |y_n - x| < \varepsilon \} \\ &= \{ h(n) : |x_{h(n)} - x| < \varepsilon \} \\ &= \{ n : |x_n - x| < \varepsilon \} = X_\varepsilon \end{aligned}$$

and h is a bijection then $Y_\varepsilon \sim X_\varepsilon$ for every $\varepsilon > 0$.

It follows that x is a cluster point of y_n.

Thus $x \in F$ and so $E \subseteq F$.

Since h is a bijection then h^{-1} is a bijection and $x_n = y_{h^{-1}(n)}$ for all $n \in \mathbb{N}$ then a similar argument shows that $F \subseteq E$. Thus $E = F$.