Math 713 Take Home Quiz 2 Version A

Work 5 out of the following 8 problems for full credit. Clearly indicate which problems you have solved and use separate sheets of paper for each solution. Do not place more than one solution on a single sheet of paper.

It is okay to consult books, journals and internet links. Do not hide the books in the library from your classmates and do not post questions to online help forums. It is okay to share books, page numbers and internet links with your classmates but do not discuss the questions with any other person except me. Behaviors inappropriate to test taking will be considered cheating. Please give detailed references for any sources outside the class notes and class textbook that helped you solve the problems.

1. Let $E \in \mathcal{M}$ and $0 < a < \lambda(E)$. Prove or disprove the claim that there exists a closed set $F \subseteq E$ such that $\lambda(F) = a$.

2. Let $A, B \in \mathcal{M}$ and $A + B = \{\, a + b : a \in A \text{ and } b \in B \,\}$. Prove or disprove the claim that $A + B \in \mathcal{M}$.

3. Let $f\colon [1, \infty) \to \mathbf{R}$ be a continuous function. Prove or disprove the claim that there exists a sequence of polynomials p_n such that $p_n \to f$ uniformly on $[1, \infty)$.

4. Let $a, b \in \mathbf{R}$ and $f\colon (a, b) \to \mathbf{R}$ be uniformly continuous. Prove or disprove the claim that f is bounded.

5. Let $f\colon \mathbf{R} \to \mathbf{R}$ be Lebesgue measurable and define $E = \{\, a : \lim_{x \to a} f(x) = f(a) \,\}$. Prove or disprove the claim that E is an open set.

6. Suppose f_n is a sequence of nonnegative Lebesgue measurable functions such that $f_n \to f$ pointwise and $\int_0^1 f_n \to L$ as $n \to \infty$. Prove or disprove that $\int_0^1 f = L$.

7. Let f be a nonnegative Lebesgue measurable function such that $\int f < \infty$. Let E_n be a monotone sequence of Lebesgue measurable sets such that $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$. Define $E = \bigcap_{n=1}^{\infty} E_n$. Prove or disprove the claim that $\int_E f = \lim_{n \to \infty} \int_{E_n} f$.

8. For each $n \in \mathbf{N}$ let $E_n = [-2^n, 2^n]$ and define $f\colon \mathbf{R} \to \mathbf{R}$ by

$$f = \sum_{n=1}^{\infty} e^{-n} \chi_{E_n}.$$

Show that f is a well-defined nonnegative $\mathcal{M}$-measurable function and use the Monotone Convergence Theorem to evaluate $\int f$.