Quiz 2 Solutions

#1. Let $E \in \mathcal{M}$ and $0 < \alpha < \lambda(E)$. Prove or disprove the claim that there exists a closed set $F \subseteq E$ such that $\lambda(F) = \alpha$.

Define $E_n = [-n, n] \cap E$. Then $E_1 \subseteq E_2 \subseteq \cdots$ so by Theorem 3.13

$$\lambda(E) = \lambda\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n \to \infty} \lambda(E_n)$$

It follows that there is some n_0 large enough so that

$$0 < \alpha < \lambda(E_{n_0}) \leq 2n_0 < \infty.$$

Let $\varepsilon_1 = \lambda(E) - \alpha$. By problem 3.46 there is a closed set $F_1 \subseteq E_{n_0}$ such that $\lambda(E_{n_0}) < \lambda(F_1) + \varepsilon_1$. Now

$$\alpha = \lambda(E_{n_0}) - (\lambda(E) - \alpha) = \lambda(E_{n_0}) - \varepsilon_1 < \lambda(F_1)$$

Define $f(x) = \lambda(F_1 \cap (-\infty, x])$.

Claim $f : \mathbb{R} \to [0, 2n_0]$ is continuous.

Clearly $0 \leq f(x) = \lambda(F_1 \cap (-\infty, x]) \leq \lambda(F_1) \leq \lambda(E_{n_0}) \leq 2n_0$ so the function f is well defined. For $\varepsilon > 0$ arbitrary choose $\delta = \varepsilon$. Then $0 < |x_1 - x_2| < \delta$ with $x_1 < x_2$ implies

$$|f(x_1) - f(x_2)| = |\lambda(F_1 \cap (-\infty, x_1]) - \lambda(F_1 \cap (-\infty, x_2])|$$

$$= \lambda(F_1 \cap (x_1, x_2]) \leq \lambda((x_1, x_2]) \leq |x_2 - x_1| < \delta = \varepsilon.$$

Since $f(n_0) = \lambda(F_1 \cap (-\infty, n_0]) = \lambda(F_1) > \alpha$

and $f(-2n_0) = \lambda(F_1 \cap (-\infty, -2n_0)) = \lambda(\emptyset) = 0 < \alpha$, then by the intermediate value theorem for continuous functions there is a point $c \in (-2n_0, n_0)$ such that $f(c) = \alpha$.

Define $F = F_1 \cap (-\infty, c]$. Then F is closed, $F \subseteq F_1 \subseteq E_{n_0} \subseteq E$ and $\lambda(F) = \alpha$.

3.45. Let $E \in \mathcal{M}$. Then $\lambda(E) = \inf\{\lambda(O) : E \subseteq O$ and O is open$\}$.

Since $E \subseteq O$ implies $\lambda(E) \leq \lambda(O)$ then

$$\lambda(E) \leq \inf\{\lambda(O) : E \subseteq O \text{ and } O \text{ is open}\}.$$

Let $\varepsilon > 0$. By definition of $\lambda(E)$ there exists open intervals I_n such that $E \subseteq \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} \ell(I_n) < \lambda(E) + \varepsilon$.

Define $O = \bigcup_{n=1}^{\infty} I_n$. Then O is open and $E \subseteq O$ and

$$\lambda(O) \leq \sum_{n=1}^{\infty} \lambda(I_n) = \sum_{n=1}^{\infty} \ell(I_n) < \lambda(E) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary it follows that

$$\inf\{\lambda(O) : E \subseteq O \text{ and } O \text{ is open}\} \leq \lambda(E)$$

Therefore $\lambda(E) = \inf\{\lambda(O) : E \subseteq O$ and O is open$\}$.

3.46 Let $E \in \mathcal{M}$. Then $\lambda(E) = \sup\{\lambda(F) : F \subseteq E$ and F is closed$\}$.

Since $F \subseteq E$ implies $\lambda(F) \le \lambda(E)$ then

$$\lambda(E) \ge \sup\{\lambda(F) : F \subseteq E \text{ and } F \text{ is closed}\}.$$

Define $E_n = [-n, n] \cap E$. Since $E_1 \subseteq E_2 \subseteq \cdots$ then it follows from Theorem 3.13 that $\lambda(E) = \lambda(\bigcup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \lambda(E_n)$.

Case $\lambda(E) < \infty$. Let $\varepsilon > 0$ be arbitrary and choose n_0 so large that $\lambda(E) < \lambda(E_{n_0}) + \varepsilon/2$.

By the ~~previous~~ exercise there is open O such that $[-n_0, n_0] \setminus E_{n_0} \subseteq O$ and $\lambda(O) \le \lambda([-n_0, n_0] \setminus E_{n_0}) + \varepsilon/2$. Since $\lambda([-n_0, n_0] \setminus E_{n_0}) < \infty$ it follows that $\lambda(O \setminus ([-n_0, n_0] \setminus E_{n_0})) < \varepsilon/2$.

Let $F = [-n_0, n_0] \setminus O$. Then

$$E_{n_0} \setminus F = E_{n_0} \setminus ([-n_0, n_0] \setminus O) = E_{n_0} \cap ([-n_0, n_0] \cap O^c)^c = E_{n_0} \cap ([-n_0, n_0]^c \cup O)$$
$$= (E_{n_0} \cap [-n_0, n_0]^c) \cup (E_{n_0} \cap O) = E_{n_0} \cap O \subseteq O \cap ([-n_0, n_0]^c \cup E_{n_0})$$
$$= O \cap ([-n_0, n_0] \cap E_{n_0}^c)^c = O \setminus ([-n_0, n_0] \setminus E_{n_0})$$

implies $\lambda(E_{n_0} \setminus F) \le \lambda(O \setminus ([-n_0, n_0] \setminus E_{n_0})) < \varepsilon$. Since $\lambda(F) \le 2n_0 < \infty$ it follows that $\lambda(E_{n_0}) < \lambda(F) + \varepsilon/2$.

Therefore $\lambda(E) < \lambda(E_{n_0}) + \varepsilon/2 < \lambda(F) + \varepsilon/2 + \varepsilon/2 = \lambda(F) + \varepsilon$.

Since $\varepsilon > 0$ was arbitrary then $\lambda(E) \le \sup\{\lambda(F) : F \subseteq E$ and F is closed$\}$.
It follows that $\lambda(E) = \sup\{\lambda(F) : F \subseteq E$ and F is closed$\}$.

Case $\lambda(E) = \infty$. Let $M > 0$ and choose n_0 so large that $M < \lambda(E_{n_0})$.
Choose F as in the previous case. Then $M < \lambda(E_{n_0}) < \lambda(F) + \varepsilon/2$
Implies $\lambda(F) \ge M - \varepsilon/2$.

Since M was arbitrary it follows that

$$\sup\{\lambda(F) : F \subseteq E \text{ and } F \text{ is closed}\} = \infty.$$

#2. Let $A, B \in \mathcal{M}$ and $A+B = \{a+b : a \in A \text{ and } b \in B\}$. Prove or disprove the claim that $A+B \in \mathcal{M}$.

The claim is false. See for example pages 48-49 in "Counter Examples in Probability and Real analysis" by Wise and Hall.

#3. Let $f: [1,\infty) \to \mathbb{R}$ be a continuous function. Prove or disprove the claim that there exists a sequence of polynomials p_n such that $p_n \to f$ uniformly on $[1,\infty)$.

<u>The claim is false</u>: Let $f(x) = \frac{1}{x}$. Then f is continuous and $\lim_{x\to\infty} f(x) = 0$.

Suppose there were a sequence of polynomials that uniformly converged to $\frac{1}{x}$. Then for $\varepsilon = \frac{1}{7}$ there is n_0 large enough that $|p_{n_0}(x) - \frac{1}{x}| < \frac{1}{7}$ for all $x \in [1,\infty)$.

Clearly $p_{n_0}(x)$ can't be the constant function since there is no constant c such that $|c - \frac{1}{x}| < \frac{1}{7}$ for all $x \in [1,\infty)$. Therefore $P_{n_0}(x)$ must have degree 1 or greater.

But then $\lim_{x\to\infty} |P_{n_0}(x)| = \infty$ again contradicting that $|P_{n_0}(x) - \frac{1}{x}| < \frac{1}{7}$ for all $x \in [1,\infty)$.

Thus there is no sequence of polynomials that converge uniformly on $[1,\infty)$ to $1/x$.

#4, Let $a, b \in \mathbb{R}$ and $f:(a,b) \to \mathbb{R}$ be uniformly continuous. Prove or disprove the claim that f is bounded.

<u>Proof</u>: Since f is uniformly continuous, there exists $\delta > 0$ such that $x_1, x_2 \in (a,b)$ with $|x_1 - x_2| < \delta$ implies $|f(x_1) - f(x_2)| < 1$.

Let $x_0 = \frac{a+b}{2}$ and choose N so large that $\frac{b-a}{2N} < \delta$.

Define $M = f(x_0) + N$.

<u>Claim</u> $|f(x)| \leq M$ for every $x \in (a,b)$.

Let $x \in (a,b)$. Define $x_k = x_0 + k\frac{b-a}{2N}$. Let k be an integer so that $x \in (x_k, x_{k+1})$. Since $x_{-N} = a$ and $x_N = b$ we must have that $-N \leq k \leq N-1$.

Case $k=0$. Then

$$|f(x)| \leq |f(x_0)| + |f(x_0) - f(x)| \leq |f(x_0)| + 1 \leq M.$$

Case $k>0$. Then

$$|f(x)| \leq |f(x_0)| + \sum_{n=0}^{k-1} |f(x_n) - f(x_{n+1})| + |f(x_k) - f(x)|$$

$$\leq |f(x_0)| + (k-1) + 1 \leq |f(x_0)| + N - 1 \leq M.$$

Case $k=-1$. Then

$$|f(x)| \leq |f(x_0)| + |f(x_0) - f(x)| \leq |f(x_0)| + 1 \leq M$$

Case $k<-1$. Then

$$|f(x)| \leq |f(x_0)| + \sum_{n=k+1}^{-1} |f(x_{n+1}) - f(x_n)| + |f(x_{k+1}) - f(x)|$$

$$\leq |f(x_0)| + |k+1| + 1 \leq |f(x_0)| + N \leq M.$$

#5. Let $f: \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable and define

$$E = \{a : \lim_{x \to a} f(x) = f(a)\}.$$

Prove or disprove the claim that E is an open set.

Define

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{|q|} & \text{if } x = \frac{p}{q} \text{ in lowest terms.} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Then f is continuous at every irrational number and discontinuous at every rational number.

Thus $E = \mathbb{R} \setminus \mathbb{Q}$ which is not open

Claim f is measurable

Let $\frac{p_n}{q_n}$ be an enumeration of the rationals expressed in lowest terms where 0 is written $\frac{0}{1}$.

Define

$$g_n = \frac{1}{|q_n|} \chi_{\{\frac{p_n}{q_n}\}}$$

Since the singleton set $\{\frac{p_n}{q_n}\} \in \mathcal{B}$ then $g_n \in \hat{C} \subseteq \mathcal{L}$. Since $\hat{C}$ is an algebra then

$$f_N = \sum_{n=1}^{N} g_n \in \hat{C} \subseteq \mathcal{L},$$

Since $\hat{C}$ is closed under pointwise limits, then

$$f = \lim_{N \to \infty} f_N \in \hat{C} \subseteq \mathcal{L}.$$

#6. Suppose f_n is a sequence of non-negative Lebesgue measurable functions such that $f_n \to f$ pointwise and $\int_0^1 f_n \to L$ as $n \to \infty$. Prove or disprove that $\int_0^1 f = L$.

Define $f_n = n \chi_{(0, \frac{1}{n})}$

Then $f_n \in \mathcal{Y}$ and moreover

$$\int f_n = n \lambda((0, \tfrac{1}{n})) = 1 \quad \text{for all } n \in \mathbb{N}$$

therefore

$$\lim_{n \to \infty} \int f_n = 1.$$

However

$$f_n \to 0 \quad \text{pointwise as } n \to \infty$$

and

$$\int 0 = 0.$$

#7. Let f be a non-negative Lebesgue measurable function such that $\int f < \infty$. Let E_n be a monotone sequence of Lebesgue measurable sets such that $E_1 \supseteq E_2 \supseteq \cdots$. Define $E = \bigcap_{n=1}^{\infty} E_n$. Prove or disprove the claim that $\int_E f = \lim_{n\to\infty} \int_{E_n} f$.

Define $f_n(x) = f(x)\chi_{E_n}(x)$.

Since $E_1 \supseteq E_2 \supseteq \cdots$ then $f \geq f_1 \geq f_2 \geq \cdots$

Furthermore $\int f_1 \leq \int f < \infty$ by the monotonicity property of the integral.

Since $f_n \to \chi_E f$ pointwise then Theorem 3.19 implies

$$\lim_{n\to\infty} \int f_n = \int \chi_E f$$

Therefore

$$\lim_{n\to\infty} \int_{E_n} f = \lim_{n\to\infty} \int f_n = \int \chi_E f = \int_E f.$$

#8 For each $n \in \mathbb{N}$ define $E_n = [-2^n, 2^n]$ and define $f: \mathbb{R} \to \mathbb{R}$ by

$$f = \sum_{n=1}^{\infty} e^{-n} \chi_{E_n}$$

Show that f is well defined nonnegative $\mathcal{M}$-measurable function and use the Monotone Convergence Theorem to evaluate $\int f$.

(i) Well defined. It is enough to show that the series converges for every value of x.

Let $x \in \mathbb{R}$. Then $x \in E_n$ for n so large $|x| \leq 2^n$. It follows that

$$\sum_{n=1}^{\infty} e^{-n} \chi_{E_n}(x) = \sum_{|x| \leq 2^n} e^{-n} = \sum_{n=m}^{\infty} e^{-n} = \frac{e^{-m}}{1-e^{-1}} < \infty.$$

where m is the integer such that $2^{m-1} < |x| \leq 2^m$.

(ii) Non-negative f is nonnegative because it is a sum of nonnegative functions

(iii) $\mathcal{M}$-measurable since $E_n \in \mathcal{M}$ then $\chi_{E_n} \in \mathcal{L}$ and since $\mathcal{L}$ is an algebra then $f_N = \sum_{n=1}^{N} e^{-n} \chi_{E_n} \in \mathcal{L}$. Furthermore $\mathcal{L}$ is closed under pointwise limits, therefore $f \in \mathcal{L}$.

(iv) Evaluate $\int f$ using the monotone convergence theorem.

The monotone convergence theorem and in particular the corollary given by Theorem 3.18 implies that

$$\int f = \sum_{n=1}^{\infty} e^{-n} \int \chi_{E_n} = \sum_{n=1}^{\infty} e^{-n} 2^{n+1}$$

$$= 2 \sum_{n=1}^{\infty} \left(\frac{2}{e}\right)^n = 2 \frac{2/e}{1-2/e} = \frac{4}{e-2}$$