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Math 714: Assignment #2

3.32. *Verify that if $A \in \mathcal{M}$, $\lambda(A) = 0$, and $B \subset A$, then $B \in \mathcal{M}$ and $\lambda(B) = 0$.*

Suppose that $A \in \mathcal{M}$ with $\lambda(A) = 0$, and let B be any subset of A. By the nonnegativity and monotonicity of Lebesgue outer measure, we have

$$0 \leq \lambda^*(B) \leq \lambda^*(A) = 0,$$

and so $\lambda^*(B) = 0$. Using Proposition 3.4 from our text, we conclude that $B \in \mathcal{M}$. Moreover, we have

$$\lambda(B) = \lambda^*(B) = 0,$$

as claimed. ■

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3.44. *A set is called a G_δ-set if it is the intersection of a countable number of open sets; and a set is called an F_σ-set if it is the union of a countable number of closed sets. Note that G_δ-sets and F_σ-sets are Borel sets. Now suppose that $E \in \mathcal{M}$.*
(a) *Show that there is a G_δ-set, G, and an F_σ-set, F, such that $F \subset E \subset G$ and $\lambda(E \setminus F) = \lambda(G \setminus E) = 0$.*
(b) *Referring to part* (a)*, deduce that $\lambda(F) = \lambda(E) = \lambda(G)$.*

We begin by proving the following claim: for each $\epsilon > 0$, there is an open set, O, with $E \subset O$ and $\lambda(O \setminus E) < \epsilon$. First suppose that $\lambda(E) < \infty$. We have

$$\lambda(E) = \lambda^*(E) = \inf \left\{ \sum_{n=1}^{\infty} l(I_n) : \{I_n\}_{n=1}^{\infty} \text{ open intervals}, E \subset \bigcup_{n=1}^{\infty} I_n \right\},$$

and so $\lambda(E) + \epsilon$ is not a lower bound for $\{\sum_{n=1}^{\infty} l(I_n) : \{I_n\}_{n=1}^{\infty} \text{ open intervals}, E \subset \bigcup_{n=1}^{\infty} I_n\}$. This means that there is some sequence $\{J_n\}_{n=1}^{\infty}$ of open intervals such that $E \subset \bigcup_{n=1}^{\infty} J_n$ and $\sum_{n=1}^{\infty} l(J_n) < \lambda(E) + \epsilon$. If we put $O = \bigcup_{n=1}^{\infty} J_n$, then O is open and $E \subset O$. We also have

$$\lambda(O) \leq \sum_{n=1}^{\infty} \lambda(J_n) = \sum_{n=1}^{\infty} l(J_n) < \lambda(E) + \epsilon,$$

and so, since $\lambda(E)$ is finite, $\lambda(O \setminus E) = \lambda(O) - \lambda(E) < \epsilon$. This proves the claim for $E \in \mathcal{M}$ with finite measure.

Now let $E \in \mathcal{M}$ be arbitrary, and fix $\epsilon > 0$. Put $E_n = \{x \in E : |x| < n\} = E_n \cap (-n, n)$ for each natural number n, so that $\lambda(E_n) \leq 2n < \infty$ and $E = \bigcup_{n=1}^{\infty} E_n$. By what we showed above, for each n we can find an open set O_n such that $E_n \subset O_n$ and $\lambda(O_n \setminus E_n) < \epsilon/2^n$. Let $O = \bigcup_{n=1}^{\infty} O_n$. Then O is open, $E \subset O$, and

$$\begin{aligned}
\lambda(O \setminus E) &= \lambda\left(\bigcup_{n=1}^{\infty} O_n \setminus \bigcup_{n=1}^{\infty} E_n\right) \\
&= \lambda\left(\bigcup_{n=1}^{\infty} (O_n \setminus E_n)\right) \\
&\leq \sum_{n=1}^{\infty} \lambda(O_n \setminus E_n) \\
&< \sum_{n=1}^{\infty} \epsilon/2^n \\
&= \epsilon,
\end{aligned}$$

completing the proof of the claim.

For each $n \in \mathcal{N}$, let G_n be an open set containing E such that $\lambda(G_n \setminus E) < 1/n$. The existence of such a set is guaranteed by the claim proved above. Put $G = \cap_{n=1}^{\infty} G_n$. Then G is a G_δ-set and $E \subset G$. We have

$$\lambda(G \setminus E) \leq \lambda(G_n \setminus E) < 1/n$$

for all n, so we conclude that $\lambda(G \setminus E) = 0$.

Note that $E^c \in \mathcal{M}$, so the claim above allows us to find, for each $n \in \mathcal{N}$, an open set O_n such that $E^c \subset O_n$ and $\lambda(O_n \setminus E^c) < 1/n$. Let $F_n = O_n^c$. Then F_n is closed (since O_n is open), and $E^c \subset O_n$ implies that $F_n \subset E$. Since $E \setminus F_n = E \cap O_n = O_n \cap E = O_n \setminus E^c$, we have $\lambda(E \setminus F_n) = \lambda(O_n \setminus E^c) < 1/n$. Now let $F = \bigcup_{n=1}^{\infty} F_n$, so that F is an F_σ-set with $F \subset E$. We have

$$\lambda(E \setminus F) \leq \lambda(E \setminus F_n) < 1/n$$

for each n, and so $\lambda(E \setminus F) = 0$.

Finally, observe that since $F \subset E$, we have

$$\lambda(E) = \lambda(F) + \lambda(E \setminus F) = \lambda(F) + 0 = \lambda(F).$$

Similarly, since $E \subset G$, we have

$$\lambda(G) = \lambda(E) + \lambda(G \setminus E) = \lambda(E) + 0 = \lambda(E).$$

Hence $\lambda(F) = \lambda(E) = \lambda(G)$. ■

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4.16. *Prove that the measure space, $(\mathcal{R}, \mathcal{M}, \lambda)$, is the completion of the measure space, $(\mathcal{R}, \mathcal{B}, \lambda_{|\mathcal{B}})$. Use the following steps:*
(a) *Verify that $\overline{\mathcal{B}} \subset \mathcal{M}$ by employing Exercise 3.32.*
(b) *Show that $\overline{\mathcal{B}} \supset \mathcal{M}$ by applying Exercise 3.44.*
(c) *Prove that $\lambda = \overline{\lambda_{|\mathcal{B}}}$.*

Let $B \in \overline{\mathcal{B}}$. That is, suppose we can write $B = A \cup C$, where $A \in \mathcal{B}$ and C is a subset of some Borel set with Lebesgue measure 0. By Exercise 3.32, $C \in \mathcal{M}$. Since $A \in \mathcal{B} \subset \mathcal{M}$ and $C \in \mathcal{M}$, the union $B = A \cup C$ must be in $\mathcal{M}$ as well, $\mathcal{M}$ being closed under unions. This shows that $\overline{\mathcal{B}} \subset \mathcal{M}$.

Now let $E \in \mathcal{M}$. Choose $F \in \mathcal{B}$ and $G \in \mathcal{B}$ as in Exercise 3.44 so that $F \subset E \subset G$, $\lambda(E \setminus F) = \lambda(G \setminus E) = 0$, and $\lambda(F) = \lambda(E) = \lambda(G)$. Note that $G \setminus F = G \cup F^c \in \mathcal{B}$ since $\mathcal{B}$ is closed under complements and unions, and that

$$\lambda(G \setminus F) = \lambda\big((G \setminus E) \cup (E \setminus F)\big) = \lambda(E \setminus F) + \lambda(G \setminus E) = 0.$$

This means that $E \setminus F$ is a subset of a Borel set with measure 0. We have $E = F \cup (E \setminus F)$, and so E is the union of a Borel set and a subset of a Borel set with measure 0. This means that $E \in \overline{\mathcal{B}}$, and consequently we conclude that $\overline{\mathcal{B}} \supset M$.

Having proved that $\mathcal{M} = \overline{\mathcal{B}}$, it remains to show that $\lambda = \overline{\lambda_{|\mathcal{B}}}$. To this end, suppose $E \in \mathcal{M} = \overline{\mathcal{B}}$. We can write $E = A \cup C$, where $A \in \mathcal{B}$ and C is a subset of some Borel set with Lebesgue measure 0 (so $\lambda(C) = 0$). We have $A \subset E$, hence $\lambda(A) \leq \lambda(E)$, and

$$\lambda(E) = \lambda(A \cup C) \leq \lambda(A) + \lambda(C) = \lambda(A) + 0.$$

It follows that $\lambda(E) = \lambda(A)$. Since $\overline{\lambda_{|\mathcal{B}}}(E) = \lambda(A)$ by definition, we have $\lambda(E) = \overline{\lambda_{|\mathcal{B}}}(E)$, completing the proof. ■

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4.80. *Suppose that $f \in \mathcal{L}^1(\Omega, \mathcal{A}, \mu)$. Show that for each $\epsilon > 0$, there is a $\delta > 0$ such that $\mu(E) < \delta$ implies $\int_E |f| \, d\mu < \epsilon$.*

Define a nondecreasing sequence $\{f_n\}_{n=1}^{\infty}$ of nonnegative functions by

$$f_n(x) = \begin{cases} |f(x)| & \text{if } |f(x)| \leq n; \\ n & \text{otherwise} \end{cases}$$

for each $n \in \mathcal{N}$. This sequence converges pointwise to $|f|$, so by the Monotone Convergence Theorem,

$$\lim_{n \to \infty} \int_\Omega f_n \, d\mu = \int_\Omega |f| \, d\mu.$$

Since $\int_\Omega f_n \, d\mu \le \int_\Omega |f| \, d\mu < \infty$ for all n, this means that we can choose an n large enough to ensure

$$\int_\Omega |f| \, d\mu - \int_\Omega f_n \, d\mu = \int_\Omega (|f| - f_n) \, d\mu < \epsilon/2. \tag{1}$$

Next, note that $f_n \le n$, and so $\int_E f_n \, d\mu \le \int_E n \, d\mu = n\mu(E)$ for any $E \in \mathcal{A}$. Put $\delta = \epsilon/2n$. Then $\mu(E) < \delta$ implies

$$\int_E f_n \, d\mu \le n\mu(E) < \epsilon/2. \tag{2}$$

It follows that if $\mu(E) < \delta$, then

$$\begin{aligned}\int_E |f| \, d\mu &= \int_E (|f| - f_n) \, d\mu + \int_E f_n \, d\mu \\ &\le \underbrace{\int_\Omega (|f| - f_n) \, d\mu}_{< \epsilon/2 \text{ by } (1)} + \underbrace{\int_E f_n \, d\mu}_{< \epsilon/2 \text{ by } (2)} \\ &< \epsilon,\end{aligned}$$

which is what we wanted to show. ■

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4.147. *Denote by $\mathcal{B}_2$ the smallest σ-algebra of subsets of $\mathcal{R}^2$ that contains all open sets of $\mathcal{R}^2$. Members of $\mathcal{B}_2$ are called two-dimensional Borel sets.*
(a) *Show that $\mathcal{B}_2 = \mathcal{B} \times \mathcal{B}$.*
(b) *A measure on $\mathcal{B}_2$ is called a two-dimensional Borel measure. Suppose that μ and ν are finite two-dimensional Borel measures such that $\mu(A \times B) = \nu(A \times B)$ for all $A, B \in \mathcal{B}$. Prove that $\mu = \nu$.*

Let us first prove that $\mathcal{B}_2$ is contained in $\mathcal{B} \times \mathcal{B}$. To do this we will use the fact that the set, $\mathcal{T}$, of open rectangles of the form $(a, b) \times (c, d)$ with $a, b, c, d \in \mathcal{Q}$, forms a countable basis for the topology on $\mathcal{R}^2$. If U is an open subset of $\mathcal{R}^2$, then we can write U as a countable union of elements of $\mathcal{T}$. But each element of $\mathcal{T}$ is of the form $(a, b) \times (c, d)$, where $(a, b) \in \mathcal{B}$ and $(c, d) \in \mathcal{B}$, so clearly elements of $\mathcal{T}$ are contained in $\mathcal{B} \times \mathcal{B}$. Hence U is a countable union of elements of $\mathcal{B} \times \mathcal{B}$, and therefore $U \in \mathcal{B} \times \mathcal{B}$. This shows that $\mathcal{B} \times \mathcal{B}$ is a σ-algebra containing all open sets of $\mathcal{R}^2$. Since $\mathcal{B}_2$ is the smallest σ-algebra containing the open sets, it follows that $\mathcal{B}_2 \subset \mathcal{B} \times \mathcal{B}$.

To show that $\mathcal{B} \times \mathcal{B} \subset \mathcal{B}_2$, we will show that every measurable rectangle is in $\mathcal{B}_2$. Consider the set $\mathcal{B}_x = \{A \times \mathcal{R} : A \in \mathcal{B}\}$ of subsets of $\mathcal{R}^2$. Note that

$$(A^c \times \mathcal{R}) = (A \times \mathcal{R})^c \qquad \text{and} \qquad \left(\bigcup_{n=1}^{\infty} A_n\right) \times \mathcal{R} = \bigcup_{n=1}^{\infty} (A_n \times \mathcal{R}).$$

Therefore, since each $A \in \mathcal{B}$ can be obtained by taking complements and countable unions of open subsets of $\mathcal{R}$, each $A \times \mathcal{R} \in \mathcal{B}_x$ can be obtained by taking complements and countable unions of sets of the form $(U \times \mathcal{R})$, where $U \subset \mathcal{R}$ is open. But each $(U \times \mathcal{R})$ is open in $\mathcal{R}^2$, and therefore is an member of $\mathcal{B}_2$, as are complements and countable unions of sets of this form. It follows that $\mathcal{B}_x \subset \mathcal{B}_2$. Similar reasoning shows that $\mathcal{B}_y = \{\mathcal{R} \times B : B \in \mathcal{B}\} \subset \mathcal{B}_2$. Now suppose that $A \in \mathcal{B}$ and $B \in \mathcal{B}$. Then $A \times B = (A \times \mathcal{R}) \cap (\mathcal{R} \times B)$ is the intersection of two members of $\mathcal{B}_2$, and so $A \times B \in \mathcal{B}_2$ since $\mathcal{B}_2$ is closed under unions. This shows that $\mathcal{B}_2$ is a σ-algebra containing the set

$$\mathcal{U} = \{A \times B : A \in \mathcal{B} \text{ and } B \in \mathcal{B}\}$$

of measurable rectangles. As $\mathcal{B} \times \mathcal{B}$ is the smallest σ-algebra containing $\mathcal{U}$, it follows that $\mathcal{B} \times \mathcal{B} \subset \mathcal{B}_2$. Having also shown the reverse containment, we now conclude that $\mathcal{B}_2 = \mathcal{B} \times \mathcal{B}$.

Now suppose that μ and ν are finite two-dimensional Borel measures such that $\mu(A \times B) = \nu(A \times B)$ for all $A, B \in \mathcal{B}$. This means that $\mu_{|\mathcal{U}} = \nu_{|\mathcal{U}}$, where

$$\mathcal{U} = \{A \times B : A \in \mathcal{B} \text{ and } B \in \mathcal{B}\}$$

is the semialgebra of measurable rectangles of $\mathcal{R}^2$. The σ-algebra generated by $\mathcal{U}$ is $\mathcal{B} \times \mathcal{B}$, which by what we showed above is equal to $\mathcal{B}_2$. Since μ and ν are finite measures, Corollary 4.7 — one of the corollaries to the uniqueness portion of the Extension Theorem — implies that $\mu = \nu$ on the measurable space $(\mathcal{R}^2, \mathcal{B}_2)$. $\blacksquare$.