Cauchy–Riemann Equations. *Let $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$. If f is a complex differentiable function then $u_x = v_y$ and $v_x = -u_y$.*

Proof. Suppose f is differentiable at z. Then the limit

$$f'(z) = \lim_{\substack{h\to 0\\ h\in\mathbf{C}}} \frac{f(z+h) - f(z)}{h}$$

exists. Since the limit exists as $h \to 0$ through all complex numbers, then it also exists as $h \to 0$ approaches through the real numbers. Therefore, the limits

$$\lim_{\substack{h\to 0\\ h\in\mathbf{R}}} \frac{f(z+h) - f(z)}{h} \qquad \text{and} \qquad \lim_{\substack{ih\to 0\\ h\in\mathbf{R}}} \frac{f(z+ih) - f(z)}{ih}$$

are both equal to $f'(z)$. Computing yields that

$$\begin{aligned}
\lim_{\substack{h\to 0\\ h\in\mathbf{R}}} \frac{f(z+h) - f(z)}{h} &= \lim_{\substack{h\to 0\\ h\in\mathbf{R}}} \frac{\big(u(x+h,y) + iv(x+h,y)\big) - \big((u(x,y) + iv(x,y)\big)}{h} \\
&= \lim_{\substack{h\to 0\\ h\in\mathbf{R}}} \frac{u(x+h,y) - u(x,y)}{h} + \lim_{\substack{h\to 0\\ h\in\mathbf{R}}} i\frac{v(x+h,y) - iv(x,y)}{h} \\
&= u_x(x,y) + iv_x(x,y)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\substack{h\to 0\\ h\in\mathbf{R}}} \frac{f(z+ih) - f(z)}{ih} &= \lim_{\substack{h\to 0\\ h\in\mathbf{R}}} \frac{\big(u(x,y+h) + iv(x,y+h)\big) - \big((u(x,y) + iv(x,y)\big)}{ih} \\
&= \lim_{\substack{h\to 0\\ h\in\mathbf{R}}} \frac{u(x,y+h) - u(x,y)}{ih} + \lim_{\substack{h\to 0\\ h\in\mathbf{R}}} \frac{v(x,y+h) - iv(x,y)}{h} \\
&= \frac{1}{i}u_y(x,y) + v_y(x,y) = -iu_y(x,y) + v_y(x,y).
\end{aligned}$$

It follows that $u_x = v_y$ and $v_x = -u_y$.

Green's Theorem. [Folland, Advanced Calculus, page 223] *Suppose S is a regular region in $\mathbf{R}^2$, that is, let S be a compact set that is the closure of its interior. Further suppose that S has a piecewise smooth boundary ∂S. If P and Q are C^1 on S then*

$$\int_{\partial S} P dx + Q dy = \int_S \big(Q_x - P_y\big)\, dx\, dy.$$

Note the path integral over ∂S is to be taken in the positive sense. This means that if

$$\partial = \cup_j \Gamma_j \qquad \text{where} \qquad \Gamma_j = \{\gamma_j(t) : t \in [0,1]\}$$

then $\gamma_j(t)$ is a one-to-one piecewise differentiable function such that as t increases the region S lies to the left of $\gamma_j(t)$.

Cauchy's Formula. *Let $R \subseteq \mathbf{C}$ be open and f be a complex differentiable function defined on R. Let Ω be open and suppose $S = \overline{\Omega} \subset R$ is a set on which Green's Theorem holds. Then*

$$\int_{\partial\Omega} f(z)dz = 0.$$

Proof. Writing $f(z) = u(x,y) + iv(x,y)$ where $z = x + iy$ we apply Green's Theorem to obtain

$$\begin{aligned}\int_{\partial\Omega} f(z)dz &= \int_{\partial\Omega} \big(u(x,y) + iv(x,y)\big)d(x+iy) \\ &= \int_{\partial\Omega} u(x,y)dx - v(x,y)dy + i\int_{\partial\Omega} v(x,y)dx + u(x,y)dy \\ &= \int_{\Omega} \big(-v_x - u_y\big)\,dx\,dy + i\int_{\Omega} \big(u_x - v_y\big)\,dx\,dy\end{aligned}$$

Since f is differentiable on R then the Cauchy–Riemann equations $u_x = v_y$ and $v_x = -u_y$ hold at every point in Ω. It follows that the last two integrals above are zero.

Lemma for Cauchy's Formula. *Let $B_\rho(z_0) = \{\, z : |z - z_0| < \rho \,\}$ be the open ball of radius ρ centered at z_0. Then*

$$\int_{\partial B_\rho(z_0)} \frac{1}{\zeta - z_0}\,d\zeta = 2\pi i.$$

Proof. Translating by z_0 we may rewrite the integral in terms of $z = \zeta - z_0$ over the set $\partial B_\rho(0)$. Since, with positive orientation

$$\partial B_\rho(0) = \{\, \gamma(t) : t \in [0, 2\pi] \,\} \qquad \text{where} \qquad \gamma(t) = \rho\cos t + i\rho\sin t,$$

then

$$\begin{aligned}\int_{\partial B_\rho(0)} \frac{1}{z}\,dz &= \int_0^{2\pi} \frac{1}{\gamma(t)}\gamma'(t)dt = \int_0^{2\pi} \frac{-\rho\sin t + i\rho\cos t}{\rho\cos t + i\rho\sin t}\,dt \\ &= \int_0^{2\pi} \frac{-\sin t + i\cos t}{\cos t + i\sin t}\cdot\frac{\cos t - i\sin t}{\cos t - i\sin t}\,dt \\ &= \int_0^{2\pi} i\frac{\cos^2 t + \sin^2 t}{\cos^2 t + \sin^2 t}\,dt = 2\pi i.\end{aligned}$$

Cauchy's Formula. *Let R, Ω and f be as in the statement of Cauchy's theorem. Then*

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}\,d\zeta \qquad \textit{for} \qquad z \in \Omega.$$

Proof. Let $\rho > 0$ be so small that $\overline{B}_\rho(z) \subset \Omega$ and define $\Omega' = \Omega \setminus \overline{B}_\rho(z) = \Omega \cap B_\rho(z)^c$ where $B_\rho(z)^c = \{\, \zeta : |\zeta - z| > \rho \,\}$ is the complement of $\overline{B}_\rho(z)$. It follows that

$$\partial\Omega' = \partial\Omega \cup \partial B_\rho(z)^c.$$

By Cauchy's theorem

$$\int_{\partial\Omega'} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}\,d\zeta + \int_{\partial B_\rho(z)^c} \frac{f(\zeta)}{\zeta - z}\,d\zeta = 0$$

since $z \notin \Omega'$ implies $f(\zeta)/(\zeta - z)$ is differentiable on a neighborhood of Ω'. Therefore

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}\,d\zeta &= -\frac{1}{2\pi i}\int_{\partial B_\rho(z)^c} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_{\partial B_\rho(z)} \frac{f(\zeta)}{\zeta - z}\,d\zeta \\
&= \frac{1}{2\pi i}\int_{\partial B_\rho(z)} \frac{f(z)}{\zeta - z}\,d\zeta + \frac{1}{2\pi i}\int_{\partial B_\rho(z)} \frac{f(\zeta) - f(z)}{\zeta - z}\,d\zeta \\
&= f(z) + \frac{1}{2\pi i}\int_{\partial B_\rho(z)} \frac{f(\zeta) - f(z)}{\zeta - z}\,d\zeta.
\end{aligned}$$

We claim that the last integral tends to zero as $\rho \to 0$. Since f is differentiable at z then f is continuous at z. Therefore, given $\epsilon > 0$ there is $\delta > 0$ such that $|\zeta - z| < \delta$ implies $|f(\zeta) - f(z)| < \epsilon$. Taking $\rho \le \delta$ and $\gamma(t) = z + \rho\cos(t) + \rho i \sin(t)$ yields

$$\left|\int_{\partial B_\rho(z)} \frac{f(\zeta) - f(z)}{\zeta - z}\,d\zeta\right| \le \int_0^{2\pi} \frac{|f(\gamma(t)) - f(z)|}{|\gamma(t) - z|}\big|\gamma'(t)\big|\,dt \le \int_0^{2\pi} \frac{\epsilon}{\rho}\rho\,dt = 2\pi\epsilon.$$

Since ϵ was arbitrary, we obtain

$$\lim_{\rho\to 0}\int_{\partial B_\rho(z)} \frac{f(\zeta) - f(z)}{\zeta - z}\,d\zeta = 0.$$

The result now follows.

Cauchy's Derivative Formula. *Let R, Ω and f be as in the statement of Cauchy's theorem. Then*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial\Omega} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \qquad \textit{for} \qquad z \in \Omega.$$

Proof. Since $z \in \Omega$ and Ω is open, then there exists an open set U such that $\overline{U} \subset \Omega$ and $z \in U$. Since the derivatives

$$\frac{d^n}{dz^n}\frac{1}{\zeta - z} = \frac{n!}{(\zeta - z)^{n+1}}$$

are continuous for $z \in \overline{U}$ and $\zeta \in \partial\Omega$, then the Leibniz integral rule allows us to differentiate through the integral sign in Cauchy's formula. Consequently

$$f^{(n)}(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{d^n}{dz^n}\frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{n!}{2\pi i}\int_{\partial\Omega} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \qquad \text{for} \qquad z \in \Omega.$$

Convergence of Taylor's Series. *Suppose f is differentiable on the set $B_r(z_0)$. Then*

$$f(z) = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{k!} f^{(k)}(z_0) \qquad \textit{for} \qquad z \in B_r(z_0).$$

Proof. Let $\rho < r$. Taking $R = B_r(z_0)$ and $\Omega = B_\rho(z_0)$ satisfies the hypothesis of Cauchy's formula. By the geometric series formula

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1}{\zeta - z_0}\Big/\Big(1 + \frac{\zeta - z}{\zeta - z_0} - 1\Big) = \frac{1}{\zeta - z_0}\Big/\Big(1 - \frac{z - z_0}{\zeta - z_0}\Big) \\ &= \frac{1}{\zeta - z_0}\sum_{k=0}^{\infty}\Big(\frac{z - z_0}{\zeta - z_0}\Big)^k \qquad \text{for} \qquad \Big|\frac{z - z_0}{\zeta - z_0}\Big| < 1. \end{aligned}$$

Moreover, for any $\gamma < 1$ the convergence is uniform for $z \in B_{\gamma\rho}(z_0)$ and $\zeta \in \partial B_\rho(z_0)$. Therefore, we may intechange the limit with the integral in Cauchy's formula and then apply Cauchy's derivative formula to obtain

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_{k=0}^{\infty} \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z_0}\Big(\frac{z - z_0}{\zeta - z_0}\Big)^k \\ &= \sum_{k=0}^{\infty}(z - z_0)^k \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{k!} f^{(k)}(z_0) \end{aligned}$$

for every $z \in B_\gamma \rho(z_0)$. Since the above holds for any $\gamma < 1$ and $\rho < r$, taking the limits $\rho \to r$ and $\gamma \to 1$ yields the same equality for $z \in B_r(z_0)$.

Maximal Radius of Analyticity. *As shown by problem 8 on page 37, a power series defines a differentiable function on its radius of convergence. Suppose*

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \qquad \textit{for} \qquad z \in B_r(z_0)$$

where r is the maximal radius on which the power series converges. Then, it is impossible to extend f to a differentiable function defined on $B_\rho(z_0)$ for any $\rho > r$.

Proof. For contradiction, suppose there existed $\rho > r$ and a differentiable function g defined on $B_\rho(z_0)$ such that $g(z) = f(z)$ for $z \in B_r(z_0)$. By the theorem on the convergence of Taylor series it follows that

$$g(z) = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{k!} g^{(k)}(z_0) \qquad \text{for} \qquad z \in B_\rho(z_0).$$

Since the Taylor series expanded about the point z_0 are the same for g and f then

$$a_k = \frac{g^{(k)}(z_0)}{k!}.$$

But then $\rho > r$ contradicts r being maximal. Therefore, no differentiable extension of f defined on $B_\rho(z_0)$ for $\rho > r$ could exist.

Exponential Function. *The exponential function defined by the power series*

$$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \qquad \textit{for} \qquad z \in \mathbf{C}$$

satisfies the identities

$$\frac{d}{dz} \exp(z) = \exp(z) \qquad \textit{and} \qquad \exp(z + w) = \exp(z)\exp(w).$$

Proof. By the ratio test

$$\lim_{k \to \infty} \left| \frac{z^{k+1}/(k+1)!}{z^k/k!} \right| = \lim_{k \to \infty} \frac{|z|}{k+1} = 0$$

shows the radius of convergence is ∞. From problem 8 on page 37 it follows that $\exp(z)$ is differentiable with derivative obtained using term-by-term differentiation. Thus

$$\frac{d}{dz} \exp(z) = \sum_{k=0}^{\infty} \frac{d}{dz} \frac{z^k}{k!} = \sum_{k=1}^{\infty} \frac{z^{k-1}}{(k-1)!} = \exp(z).$$

As the series is absolutely convergent we can rearrange it. Setting $m = k + l$ we obtain

$$\begin{aligned} \exp(z)\exp(w) &= \sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{l=0}^{\infty} \frac{w^l}{l!} = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{z^k}{k!} \frac{w^l}{l!} = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{z^k}{k!} \frac{w^{m-k}}{(m-k)!} \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{m} \binom{m}{k} z^k w^{m-k} = \sum_{m=0}^{\infty} \frac{1}{m!} (z + w)^m = \exp(z + w). \end{aligned}$$