Existence and Uniqueness of Solutions to ODEs

Consider the initial value problem

$$\begin{cases} y' = f(t, y) \\ y(0) = \mathbf{0}. \end{cases} \tag{IVP}$$

where $y \in \mathbf{R}^n$ and $f : D \to \mathbf{R}^n$ where $D \subseteq \mathbf{R} \times \mathbf{R}^n$ is open and contains $(0, \mathbf{0})$.

Local Existence and Uniqueness Theorem. *If f is continuous in the first variable and uniformly Lipschitz in the second, then there exists $h > 0$ and a unique function $y \in C^1([-h, h]; \mathbf{R}^n)$ that satisfies* (IVP).

Proof. Since f is uniformly Lipschitz in the second variable, there exists $\gamma > 0$ such that

$$\|f(t, y_1) - f(t, y_2)\| \le \gamma \|y_1 - y_2\|$$

for all (t, y_1) and (t, y_2) in D. Since D is open and contains $(0, \mathbf{0})$, there exists a closed rectangle $[-a, a] \times [-b, b]^n \subseteq D$. Define

$$M = \max \big\{ \|f(t, y)\| : (t, y) \in [-a, a] \times [-b, b]^n \big\},$$

$$h = \min \Big(\frac{1}{2\gamma}, \frac{b}{M}, a \Big)$$

and

$$X = \Big\{ y \in C([-h, h]; \mathbf{R}^n) : y(0) = \mathbf{0} \text{ and } \max_{t \in [-h,h]} \|y(t)\| \le b \Big\}.$$

Define $\mathcal{J} : X \to X$ by

$$\mathcal{J}(y)(t) = \int_0^t f(s, y(s)) ds$$

Claim that $\mathcal{J}$ is well defined and a contraction.

First show $\mathcal{J}$ is well defined. Let $y \in X$. Then $h \le a$ and $\|y(s)\| \le b$ for $s \in [-h, h]$ implies that $(s, y(s)) \in D$ for $s \in [-h, h]$. Thus $f(s, y(s))$ is a composition of continuous functions and therefore continuous. Moreover, its integral is also continuous. Therefore $\mathcal{J}(y) \in C([-h, h]; \mathbf{R}^n)$. Now let $t \in [-h, h]$. Since $\|y(s)\| \le b$ for all s between 0 and t it follows that $\|f(s, y(s))\| \le M$ for all s between 0 and t. Therefore

$$\begin{aligned} \|\mathcal{J}(y)(t)\| &= \Big\| \int_0^t f(s, y(s)) ds \Big\| \le \Big| \int_0^t \|f(s, y(s))\| ds \Big| \\ &\le \Big| \int_0^t M \Big| = |t| M \le hM \le b. \end{aligned}$$

Hence $\|\mathcal{J}(t)(t)\| \le b$ for all $t \in [-h, h]$. Finally noting that

$$\mathcal{J}(y)(0) = \int_0^0 f(s, y(s)) ds = \mathbf{0}$$

we conclude that $\mathcal{J}(y) \in X$.

Next show that $\mathcal{J}$ is a contraction. Let $y_1, y_2 \in X$ and $t \in [-h, h]$. Then

$$\begin{aligned}
\|\mathcal{J}(y_1)(t) - \mathcal{J}(y_2)(t)\| &= \Big\| \int_0^t \big(f(s, y_1(s)) - f(s, y_2(s))\big) ds \Big\| \\
&\le \Big| \int_0^t \big\|f(s, y_1(s)) - f(s, y_2(s))\big\| ds \Big| \\
&\le \gamma \Big| \int_0^t \big\|y_1(s) - y_2(s)\big\| ds \Big| \\
&\le \gamma h \max_{s \in [-h,h]} \|y_1(s) - y_2(s)\| \\
&\le \frac{1}{2} \max_{s \in [-h,h]} \|y_1(s) - y_2(s)\|.
\end{aligned}$$

Therefore

$$\max_{t \in [-h,h]} \|\mathcal{J}(y_1)(t) - \mathcal{J}(y_2)(t)\| \le \frac{1}{2} \max_{s \in [-h,h]} \|y_1(s) - y_2(s)\|$$

and so $\mathcal{J}$ is a contraction with constant $k = 1/2$.

Since X is a closed subset of the Banach space $C([-h, h]; \mathbf{R}^n)$ and $\mathcal{J} : X \to X$ is a contraction, then taking $y_0 = 0$ and $y_{n+1} = \mathcal{J}(y_n)$ we obtain by the contraction mapping theorem that y_n converges to the unique fixed point $y \in X$ such that $\mathcal{J}(y) = y$. The fixed point y is continuous since $X \subseteq C([-h, h], \mathbf{R}^n)$. Hence $f(s, y(s))$ is continuous. The fundamental theorem of calculus then yields that $\mathcal{J}(y)$ is differentiable. Hence y is differentiable. In particular,

$$\frac{dy(t)}{dt} = \frac{d\mathcal{J}(y)(t)}{dt} = \frac{d}{dt} \int_0^t f(s, y(s)) ds = f(t, y(t)).$$

Therefore $y' = f(t, y)$. Moreover $y(0) = \mathbf{0}$ and so y satisfies (IVP). Therefore solutions to the initial value problem (IVP) exist.

To see solutions are unique note that any solution of (IVP) is a fixed point of $y = \mathcal{J}(y)$. Since there is only one unique fixed point of $\mathcal{J}$, then we have that there is only one solution to (IVP). Hence, solutions to (IVP) are unique on the interval $[-h, h]$.