1. [Walnut Exercise 2.26] Show that if

$$\lim_{n\to\infty} a_n = a, \text{ then } \lim_{n\to\infty} \sigma_n = a \text{ where } \sigma_n = \frac{1}{n}\sum_{k=1}^{n} a_k.$$

Lemma. If $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} b_k = 0$.

Suppose $\epsilon > 0$. Since $b_n \to 0$ there is N_0 large enough such that $n \geq N_0$ implies $|b_n| < \epsilon/2$. Define $M = \sum_{k=1}^{N_0} |b_k|$ and choose $N \geq \max(N_0, 2M/\epsilon)$. Then for $n \geq N$ we have

$$\left|\frac{1}{n}\sum_{k=1}^{n} b_k\right| \leq \frac{1}{n}\sum_{k=1}^{N_0} |b_k| + \frac{1}{n}\sum_{k=N_0+1}^{n} |b_k| \leq \frac{M}{n} + \frac{1}{n}\sum_{k=N_0+1}^{n} \frac{\epsilon}{2} \leq \frac{\epsilon}{2} + \frac{n-N_0}{n}\frac{\epsilon}{2} < \epsilon.$$

This finishes the proof of the lemma.

Define $b_n = a_n - a$. Then $b_n \to 0$ and it follows from the lemma that

$$\frac{1}{n}\sum_{k=1}^{n} a_k = \frac{1}{n}\sum_{k=1}^{n}(b_k + a) = a + \frac{1}{n}\sum_{k=1}^{n} b_k \to 0 \qquad \text{as} \qquad n \to \infty.$$

2. [Walnut Exercise 2.39] Given

$$D_k(x) = \sum_{m=-k}^{k} e^{2\pi imx/a} \qquad \text{and} \qquad F_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} D_k(x),$$

prove that

$$F_n(x) = \frac{1}{n}\Big(\frac{\sin(\pi nx/a)}{\sin(\pi x/a)}\Big)^2.$$

Let $w = e^{2\pi ix/a}$. Then

$$D_k = \sum_{m=-k}^{k} w^m = \frac{w^{k+1} - w^{-k}}{w-1}$$

and therefore

$$F_n = \frac{1}{n}\sum_{k=0}^{n-1} \frac{w^{k+1} - w^{-k}}{w-1} = \frac{w^{n+1} + w^{1-n} - 2w}{n(w-1)^2}.$$

Let $\theta = \pi x/a$. Then $w = e^{2i\theta}$ and we have the identities

$$\begin{aligned} w^\alpha + w^{-\alpha} &= e^{2i\alpha\theta} + e^{-2i\alpha\theta} = 2\cos 2\alpha\theta \\ w^\alpha - w^{-\alpha} &= e^{2i\alpha\theta} - e^{-2i\alpha\theta} = 2i\sin 2\alpha\theta. \end{aligned}$$

It follows that

$$F_n = \frac{w^{n+1} + w^{1-n} - 2w}{n(w-1)^2} = \frac{w^n + w^{-n} - 2}{n(w^{1/2} - w^{-1/2})^2} = \frac{2\cos 2n\theta - 2}{n(2i\sin\theta)^2} = \frac{1-\cos 2n\theta}{2n\sin^2\theta}.$$

Applying the half-angle formula $1 - \cos 2n\theta = 2\sin^2 n\theta$ finishes the proof.

It is sometimes useful to use a program like Maple to perform all or part of an algebraic manipulation in order to avoid errors when the calculations become lengthy and tedious. The following Maple script shows how to sum the geometric series needed for the solution to the previous problem.

```
restart;
Dk:=Sum(w^m,m=-k..k):
Fn:=1/n*Sum(Dk,k=0..n-1):
Dk=simplify(eval(subs(Sum=sum,Dk)));
Fn=simplify(eval(subs(Sum=sum,Fn)));
```

The resulting output is

```
  k
-----       (k + 1)    (-k)
 \      m   w        - w
  )    w  = ----------------
 /               w - 1
-----
m = -k

n - 1 /  k      \
----- |-----    |
 \    | \      m|
  )   |  )    w |
 /    | /       |
----- |-----    |       (n + 1)    (1 - n)
k = 0 \m = -k   /      -w        - w        + 2 w
----------------- = - --------------------------
        n                                2
                              n (w - 1)
```

3. [Walnut 2.42] Prove that if $f(x)$ is continuous at $x = a$, then there is a $\delta > 0$ and a number $M > 0$ such that $|f(x)| \leq M$ for all x such that $|x - a| \leq \delta$.

Let $\epsilon = 1$. By hypothesis there is $\delta_1 > 0$ such that $|x - a| < \delta_1$ implies $|f(x) - f(a)| < 1$. Choose $M = 1 + |f(a)|$ and $\delta = \delta_1/2$. Then $|x - a| \leq \delta$ implies $|x - a| < \delta_1$ and

$$|f(x)| \leq |f(x) - f(a)| + |f(a)| < 1 + |f(a)| = M.$$

4. [Walnut 2.60] Prove that if $g_n(x)$ is an orthonormal system on an interval I and if a_n where $n = 1, \ldots, N$ is any finite sequence of numbers, then

$$\Big\| \sum_{n=1}^{N} a_n g_n \Big\|_{L^2}^2 = \sum_{n=1}^{N} |a_n|^2.$$

Since $g_n(x)$ is an orthonormal system we have that

$$\Big\langle g_n(x), g_m(x) \Big\rangle = \begin{cases} 1 & \text{for } n = m, \\ 0 & \text{otherwise.} \end{cases}$$

Also note that the paring $\langle \cdot, \cdot \rangle$ is linear in the first variable and conjugate linear in the second variable so that $\langle \alpha f, g \rangle = \alpha \langle f, g \rangle$ and $\langle f, \alpha g \rangle = \overline{\alpha} \langle f, g \rangle$. Therefore,

$$\Big\| \sum_{n=1}^{N} a_n g_n \Big\|_{L^2}^2 = \Big\langle \sum_{n=1}^{N} a_n g_n, \sum_{m=1}^{N} a_m g_m \Big\rangle = \sum_{n=1}^{N} \sum_{m=1}^{N} a_n \overline{a_m} \big\langle g_n, g_m \big\rangle = \sum_{n=1}^{N} a_n \overline{a_n}.$$

5. [Walnut 2.61] Prove that if $g_n(x)$ is any system of L^2 functions, then $\text{span}\{g_n\}$ is a linear space. That is, it is closed under the formation of linear combinations.

We need to show that $\text{span}\{g_n\}$ is closed under addition and scalar multiplication. By definition

$$\text{span}\{g_n\} = \Big\{ \sum_{k=1}^{n} a_k g_k : n \in \mathbf{N} \text{ and } a_k \in \mathbf{C} \Big\}.$$

Closed under Addition. Let $f, g \in \text{span}\{g_n\}$. Then there is $n, m \in \mathbf{N}$ and $a_k, b_k \in \mathbf{C}$ such that

$$f = \sum_{k=1}^{n} a_k g_k \qquad \text{and} \qquad g = \sum_{k=1}^{m} b_k g_k.$$

Define $N = \max(n, m)$ and

$$c_k = \begin{cases} a_k & \text{for } k > m, \\ b_k & \text{for } k > n, \\ a_k + b_k & \text{otherwise.} \end{cases}$$

Then

$$f + g = \sum_{k=1}^{N} c_k g_k \in \text{span}\{g_n\}$$

so $\text{span}\{g_n\}$ is closed under addition.

Closed under Scalar Multiplication. Let $f \in \text{span}\{g_n\}$ and $\lambda \in \mathbf{C}$. Then there is $n \in \mathbf{N}$ and $a_k \in \mathbf{C}$ such that

$$f = \sum_{k=1}^{n} a_k g_k.$$

Define $c_k = \lambda a_k$. Then

$$\lambda f = \sum_{k=1}^{n} c_k g_k \in \text{span}\{g_n\}$$

so $\text{span}\{g_n\}$ is closed under scalar multiplication.

6. [Carrier, Krook and Pearson Section 1-5 Exercise 12] Verify that a suitably defined branch of

$$f(z) = \ln\left(5 + \sqrt{\frac{z+1}{z-1}}\right)$$

is single valued in the z plane outside of a line joining the points $z = 1$ and $z = -1$. Show, however, that if one enters another sheet of the Riemann surface by crossing this line, there will be a branch point at $z = 13/12$.

The function $f(z)$ can be written as a composition of the following functions.

$$z \to \frac{z+1}{z-1}, \qquad z \to \sqrt{z}, \qquad z \to 5 + z, \qquad z \to \ln z.$$

Of these functions only $\sqrt{z}$ and $\ln z$ are multi-valued and require the choice of a branch. For $\sqrt{z}$ choose the branch given by

$$re^{i\theta} \to \sqrt{r}e^{i\theta/2} \qquad \text{where } r \geq 0 \text{ and } \theta \in (-\pi, \pi]$$

which maps $\mathbf{C} \setminus (-\infty, 0]$ onto the half plane $\{\, x + iy : x > 0 \text{ and } y \in \mathbf{R} \,\}$. Note that for real numbers $x > 0$ this branch corresponds to the usual positive square root function denoted $\sqrt{x}$. For $\ln z$ choose the principle branch, denoted as $\ln_p z$, given by

$$re^{i\theta} \to \ln r + i\theta \qquad \text{where } r > 0 \text{ and } \theta \in (-\pi, \pi]$$

which maps $\mathbf{C} \setminus (-\infty, 0]$ onto the horizontal strip $\{\, x + iy : x \in \mathbf{R} \text{ and } y \in (-\pi, \pi) \,\}$.

To show that $f(z)$ is single valued it is enough to check that the composition of functions with the branches chosen above is well defined for every point in $\mathbf{C} \setminus [-1, 1]$. This is illustrated in Figure 1 on the next page.

Notice first that the map $z \to (z + 1)/(z - 1)$ maps the real line into the real line and in particular it maps the interval $[-1, 1)$ into the ray $(-\infty, 0]$. Now,

$$w = \frac{z+1}{z-1}, \qquad w(z-1) = z + 1, \qquad wz - z = w + 1, \qquad z = \frac{w+1}{w-1}$$

shows $z \to (z + 1)/(z - 1)$ is a bijection between $\mathbf{C} \setminus \{1\}$ and $\mathbf{C} \setminus \{1\}$. This bijection maps $\mathbf{C} \setminus [-1, 1]$ onto $\mathbf{C} \setminus (-\infty, 0]$. Now $z \to \sqrt{z}$ above maps $\mathbf{C} \setminus (-\infty, 0]$ onto the right half plane which in turn is shifted to the left 5 units by $z \to z + 5$. The resulting set $\{\, x + iy : x > 5 \text{ and } y \in \mathbf{R} \,\}$ does not contain the branch line $(-\infty, 0]$ of the $\ln_p$ so $z \to \ln_p z$ is well defined on this set. It follows that $f(z)$ is well defined and therefore single valued for this choice of branches.

We now study the Riemann sheet which is entered upon crossing the line $[-1, 1]$. Crossing the line $[-1, 1]$ in the domain of $f(z)$ corresponds to crossing the branch line $(-\infty, 0]$ for the square root function. In this way we enter the negative branch of square root which shall be denoted as $-\sqrt{z}$. Now, $z \to -\sqrt{z}$ maps $\mathbf{C} \setminus (-\infty, 0]$ onto the left half plane $\{\, x + iy : x < 0 \text{ and } y \in \mathbf{R} \,\}$ which $z \to 5 + z$ maps onto $\{\, x + iy : x < 5 \text{ and } y \in \mathbf{R}, \}$. However, $\ln_p$ is not well defined on this set because it contains $(-\infty, 0]$. Therefore, additional points need to be removed from the Riemann sheet of this branch for $f(z)$ to be single valued.

To determine the points which need to be removed, take inverse images to find out which points map onto the line $(-\infty, 0]$ under the mapping

$$z \to 5 - \sqrt{\frac{z+1}{z-1}}.$$

Since $z \to 5 + z$ maps $(-\infty, -5]$ onto $(-\infty, 0]$, $z \to -\sqrt{z}$ maps $[25, \infty)$ onto $(-\infty, -5]$ and $z \to (z+1)/(z-1)$ maps $[13/12, \infty)$ onto $[25, \infty)$ we discover that the points $[13/12, \infty)$ must be removed from the Riemann sheet so that this branch of $f(z)$ is well defined. In particular there is a branch point at $13/12$ in this sheet of the Riemann surface. The composition of functions showing that this branch of $f(z)$ is well defined and single valued is illustrated in Figure 2.

Figure 1. The following cartoon shows that the branches for the positive square root $\sqrt{z}$ and principal logarithm $\ln_p z$ chosen above result in a well-defined single-valued function $f(z)$ on the on the Riemann sheet $\mathbf{C} \setminus [-1, 1]$.

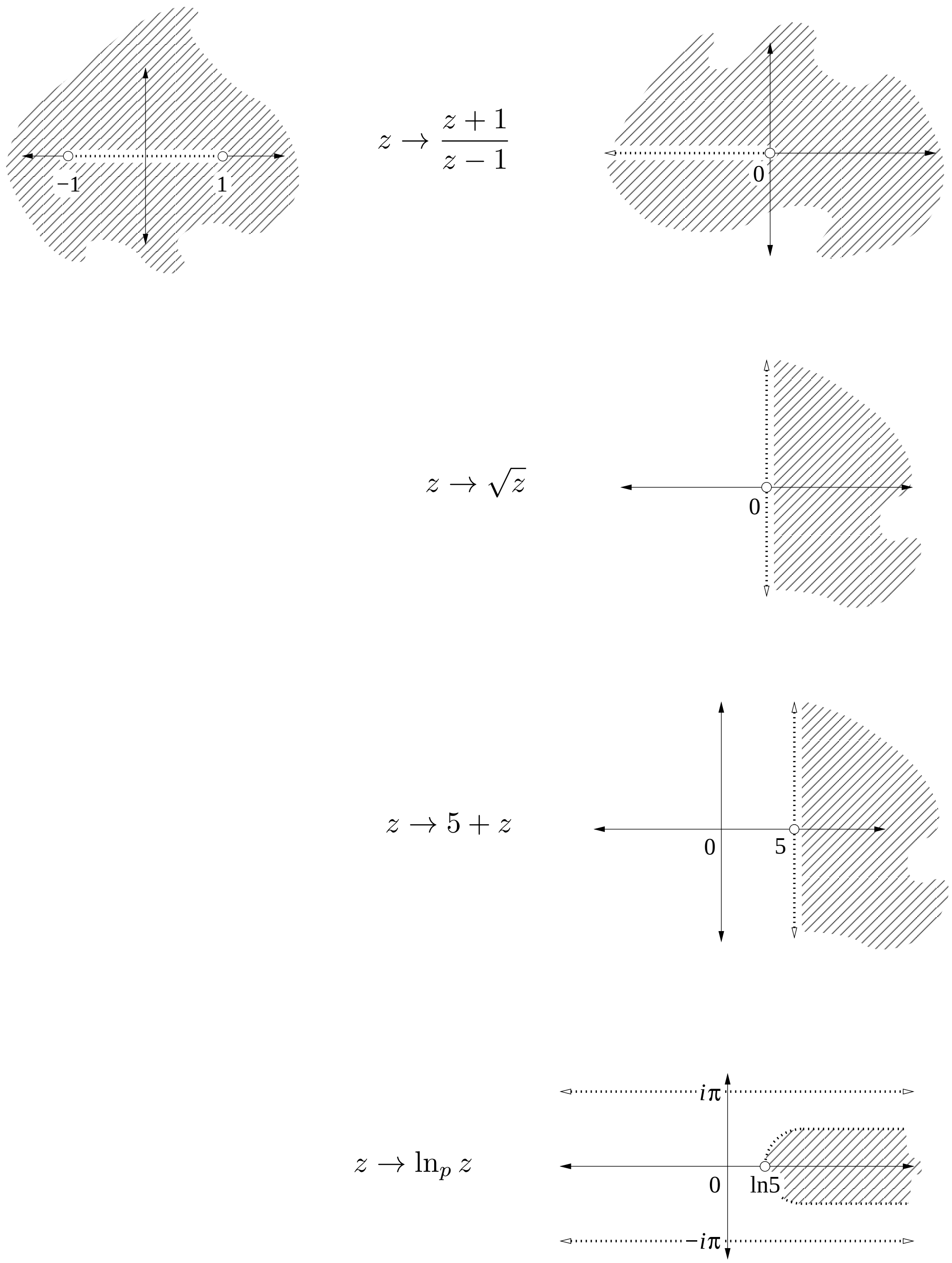
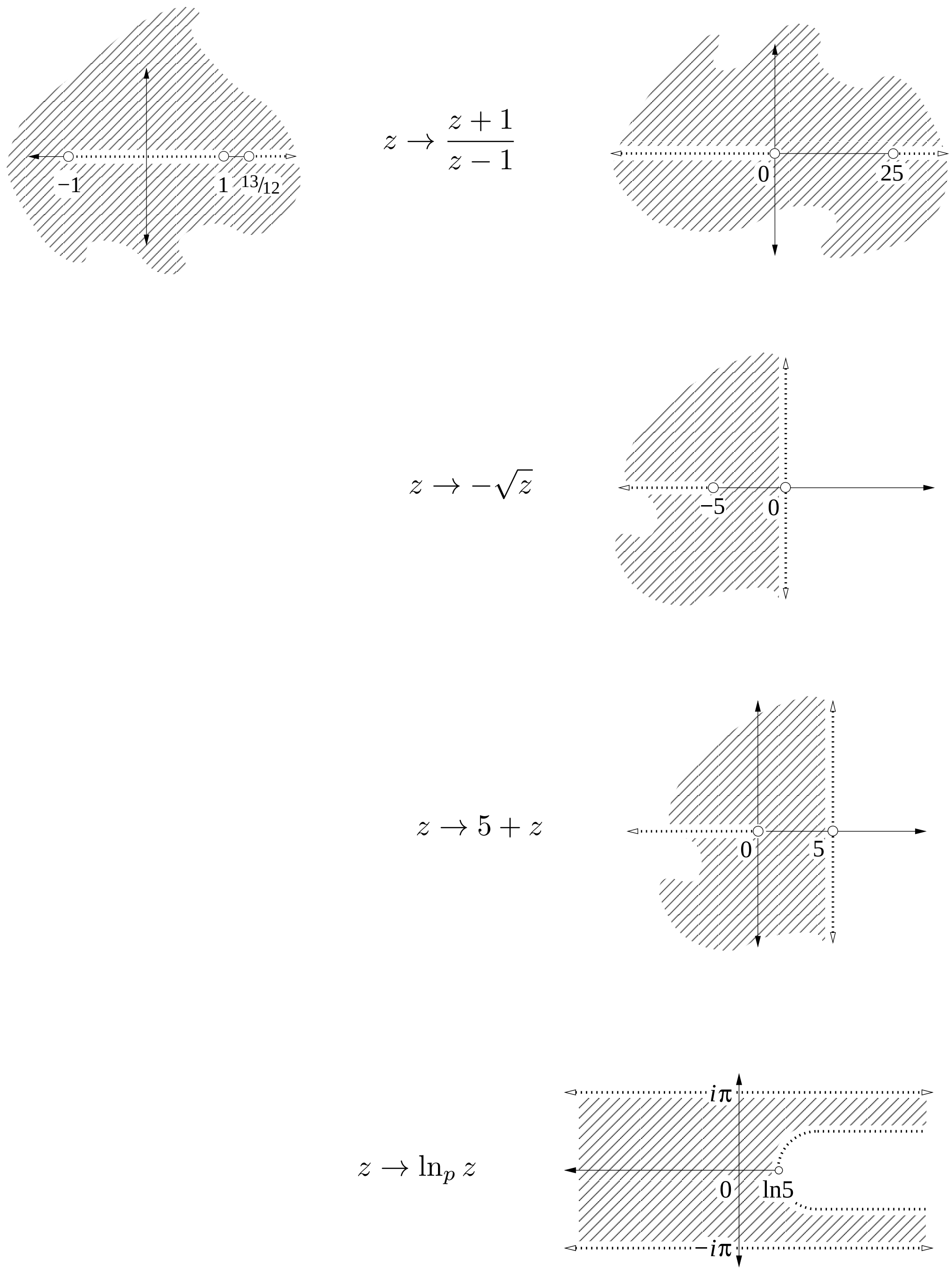


Figure 2. The following cartoon shows the branch of $f(z)$ entered by crossing the line $[-1,1]$ is well defined and single valued on the Riemann sheet $\mathbf{C} \setminus ([-1,1] \cup [13/12, \infty))$. Note that the branch point at $13/12$ is required so $\ln_p z$ is well defined in the final composition.



7. [Carrier, Krook and Pearson Section 2-1 Exercise 1] Show that no purely real function can be analytic, unless it is a constant.

Consider a function $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$ and where u and v are real functions. For f to be purely real means $v(x, y) = 0$. For f to be analytic means $u_x = v_y$ and $u_y = -v_x$ hold at every point in the complex plane. Therefore, if f is a purely-real analytic function, it follows that $u_x = 0$ and $u_y = 0$. Now $u_x = 0$ implies for y_0 fixed that the function $x \to u(x, y_0)$ is constant and similarly $u_y = 0$ implies for x_0 fixed that the function $y \to u(x_0, y)$ is constant.

Let $c = u(x_0, y_0)$. Consider any point $x_1 + iy_1$ in the complex plane. Since $x \to u(x, y_0)$ is constant holding y_0 fixed, then $u(x_1, y_0) = u(x_0, y_0) = c$. Since $y \to u(x_1, y)$ is constant holding x_1 fixed, then $u(x_1, y_1) = u(x_1, y_0) = c$. It follows that u is identically equal to c throughout the entire complex plane. Therefore f is constant.

8. [Carrier, Krook and Pearson Section 2-2 Exercise 1] Evaluate

$$\int_{1+i}^{3-2i} \sin z\, dz$$

in two ways. First by choosing any path between the two end points and using real integrals as in

$$\int_C f(z)\, dz = \int_C (u\, dx - v\, dy) + i \int_C (u\, dy + v\, dx) \tag{2-5}$$

and second by use of an indefinite integral. Show that the inequality

$$\left| \int_C f(z)\, dz \right| \leq \int_C \left| f(z) \right| |dz| \leq ML \tag{2-6}$$

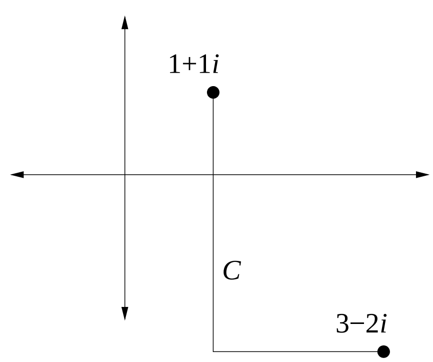
where $|f| \leq M$ and L is the length of C is satisfied.

The trigonometric identities

$$\sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$$
$$\cos z = \cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$$

will be used in this exercise.

First, consider the path C given by



which may be written as the sum of the paths $[\gamma_1]$ and $[\gamma_2]$ where $\gamma_1(t) = 1 + i(1 - 3t)$ and $\gamma_2(t) = (1 + 2t) - 2i$. Since

$$\int_C \sin z\, dz = \int_C (\sin x \cosh y\, dx - \cos x \sinh y\, dy) + i \int_C (\sin x \cosh y\, dy + \cos x \sinh y\, dx),$$

compute the real path integrals

$$\int_{[\gamma_1]} (\sin x \cosh y\, dx - \cos x \sinh y\, dy) = -\int_1^{-2} \cos 1 \sinh y\, dy = -\cos 1(\cosh 2 - \cosh 1)$$

$$\int_{[\gamma_2]} (\sin x \cosh y \, dx - \cos x \sinh y \, dy) = \int_1^3 \sin x \cosh 2 \, dx = (\cos 1 - \cos 3) \cosh 2$$

$$i \int_{[\gamma_1]} (\sin x \cosh y \, dy + \cos x \sinh y \, dx) = i \int_1^{-2} \sin 1 \cosh y \, dy = -i \sin 1(\sinh 2 + \sinh 1)$$

$$i \int_{[\gamma_2]} (\sin x \cosh y \, dy + \cos x \sinh y \, dx) = -i \int_1^3 \cos x \sinh 2 \, dx = i(\sin 1 - \sin 3) \sinh 2$$

and add them to obtain

$$\int_C \sin z \, dz = \cos 1 \cosh 1 - \cos 3 \cosh 2 - i\big(\sin 1 \sinh 1 + \sin 3 \sinh 2\big).$$

This finishes the computation using real path integrals.

Second, compute using an indefinite integral to obtain

$$\begin{aligned} \int_{1+i}^{3-2i} \sin z \, dz &= -\cos(3 - 2i) + \cos(1 + i) \\ &= -\cos 3 \cosh 2 - i \sin 3 \sinh 2 + \cos 1 \cosh 1 - i \sin 1 \sinh 1. \end{aligned}$$

This answer is the same as the answer found using the path integrals.

We now show inequality (2-6) is satisfied. The term of the left hand side is

$$\begin{aligned} \Big| \int_C \sin z \, dz \Big| &= \big| \cos(1 + i) - \cos(3 - 2i) \big| \\ &\approx |4.558275530 - 1.500720276i| \approx 4.798962091. \end{aligned}$$

To approximate the integral

$$\int_C |\sin z||dz|$$

use numerical techniques. The Maple script

```
# Compute integral |f(z)||dz| along the path gamma1+gamma2
restart;
f:=z->sin(z);
gamma1:=t->1+I*(1-3*t);
gamma2:=t->(1+2*t)-2*I;
I1:=Integrate(abs(f(gamma1(t))*diff(gamma1(t),t)),t=0..1);
F1:=evalf(I1);
I2:=Integrate(abs(f(gamma2(t))*diff(gamma2(t),t)),t=0..1);
F2:=evalf(I2);
print("The integral |f(z)||dz| is approximately", F1+F2);
```

gives the output

```
f := z -> sin(z)
```

```
gamma1 := t -> 1 + (1 - 3 t) I

gamma2 := t -> 2 t + 1 - 2 I

          1
         /
        |
I1 :=   |   3 | sin(1 + (1 - 3 t) I) | dt
        |
       /
        0

F1 := 4.468016320

          1
         /
        |
I2 :=   |   2 | sin(2 t + 1 - 2 I) | dt
        |
       /
        0

F2 := 7.429875986

"The integral |f(z)||dz| is approximately", 11.89789231
```

Therefore

$$\int_C |\sin z||dz| \approx 11.89789231.$$

Finally we compute M using Maple

```
# Find the maximum value of |f(z)| on gamma1 and gamma2
restart;
_EnvAllSolutions:=true;

f:=z->sin(z);
g:=t->1+I*(1-3*t);
ff:=f(g(t))*conjugate(f(g(t))) assuming t::real;
dff:=diff(ff,t);
cp:=solve(dff=0,t);
k:=0;
for j from 1 to nops([cp])
do
    for i from -3 to 3
    do
        c:=evalf(subs(_Z1=i,cp[j]));
        if(abs(Im(c))<0.00001 and Re(c)<1 and Re(c)>0)
        then
            k:=k+1;
```

```
            cpn[k]:=abs(c);
        end
    end
end;
cps:=[0.0,seq(cpn[n],n=1..k),1.0];
v1:=seq(abs(f(g(cps[n]))),n=1..nops(cps));
print("The maximum of |f(z)| on gamma1 is",max(v1));

g:=t->(1+2*t)-2*I;
ff:=f(g(t))*conjugate(f(g(t))) assuming t::real;
dff:=diff(ff,t);
cp:=solve(dff=0,t);
k:=0;
for j from 1 to nops([cp])
do
    for i from -3 to 3
    do
        c:=evalf(subs(_Z2=i,cp[j]));
        if(abs(Im(c))<0.00001 and Re(c)<1 and Re(c)>0)
        then
            k:=k+1;
            cpn[k]:=abs(c);
        end
    end
end;
cps:=[0.0,seq(cpn[n],n=1..k),1.0];
v2:=seq(abs(f(g(cps[n]))),n=1..nops(cps));
print("The maximum of |f(z)| on gamma2 is",max(v2));

print("The maximum of |f(z)| on C is", max(v1,v2));
print("LM is",5*max(v1,v2));
```

with the results

```
                        _EnvAllSolutions := true

                           f := z -> sin(z)

                       g := t -> 1 + (1 - 3 t) I

             ff := sin(1 + (1 - 3 t) I) sin(1 - (1 - 3 t) I)

dff := -3 I cos(1 + (1 - 3 t) I) sin(1 - (1 - 3 t) I)

     + 3 I sin(1 + (1 - 3 t) I) cos(1 - (1 - 3 t) I)

             /                              2 1/2           \
             |               -1 + (1 + tan(2) )             |
cp := -1/3 I |1 + I - arctan(---------------------) - Pi _Z1~|,
```

```
      \                        tan(2)                      /

     /                               2 1/2            \
     |                 1 + (1 + tan(2)  )              |
-1/3 I |1 + I + arctan(--------------------) - Pi _Z1~|
     \                       tan(2)                    /

                        k := 0

              cps := [0., 0.3333333333, 1.0]

       v1 := 1.445396576, 0.8414709848, 3.723196185

    "The maximum of |f(z)| on gamma1 is", 3.723196185

                g := t -> 2 t + 1 - 2 I

       ff := sin(2 t + 1 - 2 I) sin(2 t + 1 + 2 I)

dff := 2 cos(2 t + 1 - 2 I) sin(2 t + 1 + 2 I)

     + 2 sin(2 t + 1 - 2 I) cos(2 t + 1 + 2 I)

                                               2 1/2
                             1 + (1 - tanh(4)  )       Pi _Z2~
cp := -1/2 + I - 1/2 I arctanh(---------------------) + -------,
                                    tanh(4)              2

                                              2 1/2
                             -1 + (1 - tanh(4)  )       Pi _Z2~
    -1/2 + I + 1/2 I arctanh(----------------------) + -------
                                    tanh(4)              2

                        k := 0

              cps := [0., 0.2853981635, 1.0]

       v2 := 3.723196185, 3.762195691, 3.629604837

    "The maximum of |f(z)| on gamma2 is", 3.762195691

       "The maximum of |f(z)| on C is", 3.762195691

                  "LM is", 18.81097846
```

Therefore $M \approx 3.762195691$ and $L = 3 + 2 = 5$ so that $LM \approx 18.81097846$. Since

$$4.798962091 \leq 11.89789231 \leq 18.81097846$$

we have shown that inequality (2-6) is satisfied for the curve C.

9. [Carrier, Krook and Pearson Section 2-2 Exercise 8a] Show that formal, term-by-term differentiation, or integration, of a power series yields a new power series with the same radius of convergence.

Term-by-term Differentiation. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence R and $g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ have radius of convergence r. Claim $R = r$.

First show $r \geq R$. Let z be such that $0 < |z| < R$. Choose w such that $|z| < |w| < R$. Then by (1-5) on page 9 we have that the series for $f(w)$ converges absolutely. Thus,

$$\sum_{n=0}^{\infty} |a_n||w|^n < \infty.$$

Now, let $\alpha = |z|/|w|$. Then $0 < \alpha < 1$ and so $n\alpha^n \to 0$ as $n \to \infty$. It follows that $n\alpha^n$ is bounded by some constant A so that $n\alpha^n \leq A$ for all n. Now,

$$\sum_{n=1}^{\infty} n|a_n||z|^{n-1} = \frac{1}{|z|}\sum_{n=1}^{\infty} n|a_n||w|^n\alpha^n \leq \frac{A}{|z|}\sum_{n=1}^{\infty} |a_n||w|^n < \infty$$

shows that the series for $g(z)$ converges for any $|z| < R$. It follows that $r \geq R$.

Second show that $R \leq r$. Suppose $|z| < r$. Then by (1-5) on page 9 we have that the series for $g(z)$ converges absolutely. Thus,

$$\sum_{n=1}^{\infty} n|a_n||z|^{n-1} < \infty.$$

Now,

$$\sum_{n=0}^{\infty} |a_n||z|^n = |a_0| + |z|\sum_{n=1}^{\infty} |a_n||z|^{n-1} \leq |a_0| + |z|\sum_{n=1}^{\infty} n|a_n|^{n-1} < \infty.$$

shows that the series for $f(z)$ converges for any $|z| < r$. It follows that $R \leq r$.

Term-by-term Integration. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence R and $h(z) = \sum_{n=0}^{\infty} \frac{1}{n+1} a_n z^{n+1}$ have radius of convergence ρ. Claim $R = \rho$.

First show that $R \leq \rho$. Suppose $|z| < R$. Then by (1-5) on page 9 we have that the series for $f(z)$ converges absolutely. Thus,

$$\sum_{n=0}^{\infty} |a_n||z|^n < \infty.$$

Now,

$$\sum_{n=0}^{\infty} \frac{1}{n+1}|a_n||z|^{n+1} \leq |z|\sum_{n=0}^{\infty} |a_n||z|^n < \infty.$$

shows that the series for $h(z)$ converges for any $|z| < R$. It follows that $R \leq \rho$.

Second show $R \geq \rho$. Let z be such that $0 < |z| < \rho$. Choose w such that $|z| < |w| < \rho$. Then by (1-5) on page 9 we have that the series for $h(w)$ converges absolutely. Thus,

$$\sum_{n=0}^{\infty} \frac{1}{n+1} |a_n||w|^{n+1} < \infty.$$

Now, let $\alpha = |z|/|w|$. Let A be the bound so that $n\alpha^n \leq A$ for all n. Now,

$$\sum_{n=0}^{\infty} |a_n||z|^n = \frac{1}{|z|} \sum_{n=0}^{\infty} \frac{1}{n+1} |a_n||w|^{n+1}(n+1)\alpha^{n+1} \leq \frac{A}{|z|} \sum_{n=0}^{\infty} \frac{1}{n+1} |a_n||w|^{n+1} < \infty$$

shows that the series for $f(z)$ converges for any $|z| < \rho$. It follows that $R \geq \rho$.

10. [Carrier, Krook and Pearson Section 2-2 Exercise 8b] The uniform-convergence property of a power series implies that term-by-term integration yields the integral of the sum function. Show that the integrated sum function is single valued and analytic within the circle of convergence.

Let R be the radius of convergence of the infinite series defining f and h in part a. Let

$$f_n(z) = \sum_{k=0}^{n} a_k z^k \qquad \text{and} \qquad h_n(z) = \sum_{k=0}^{n} \frac{1}{k+1} a_k z^{k+1}.$$

Suppose ξ and z are such that $|\xi| < R$ and $|z| < R$. Let $[\gamma]$ be any path such that $\gamma(0) = z$, $\gamma(1) = \xi$ and $|\gamma(t)| < R$ for $t \in [0,1]$. Since f_n is a polynomial, it is analytic. Therefore the path integral along $[\gamma]$ is path independent and since $h_n' = f_n$ we obtain

$$\int_{[\gamma]} f_n(\zeta)\, d\zeta = \int_z^\xi f_n(\zeta)\, d\zeta = h_n(\xi) - h_n(z).$$

Since $|\gamma(t)| < R$ for $t \in [0,1]$ then there is $\eta > 0$ such that $|\gamma(t)| \le R - \eta$ for $t \in [0,1]$. From the results on page 9 we obtain that

$$f_n\big(\gamma(t)\big) \to f\big(\gamma(t)\big) \qquad \text{uniformly in } t \text{ as } n \to \infty.$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} \int_{[\gamma]} f_n(\zeta)\, d\zeta &= \lim_{n\to\infty} \int_0^1 f_n(\gamma(t))\gamma'(t)\, dt \\ &= \int_0^1 f(\gamma(t))\gamma'(t)\, dt = \int_{[\gamma]} f(\zeta)\, d\zeta. \end{aligned}$$

It follows that

$$\int_{[\gamma]} f(\zeta)\, d\zeta = \lim_{n\to\infty} \big(h_n(\xi) - h_n(z)\big) = h(\xi) - h(z).$$

Since this equality holds for any $[\gamma]$ inside the radius of convergence, the integral is path independent. Therefore, we may write

$$\int_z^\xi f(\zeta)\, d\zeta = h(\xi) - h(z)$$

for any z and ξ such that $|\xi| < R$ and $|z| < R$ where the integral is to be interpreted as a path integral along along any path from z to ξ that lies strictly inside the radius of convergence. Since h is single valued then the integrated sum function is single valued.

We now claim h is analytic and $h'(z) = f(z)$. Let $\epsilon > 0$. Since f is continuous at z there is $\delta > 0$ so that $|\zeta - z| < \delta$ implies $|f(\zeta) - f(z)| < \epsilon$. Define $\gamma(t) = \xi(1-t) + zt$ so that $\gamma'(t) = z - \xi$. Now, $|\xi - z| < \delta$ implies $|\gamma(t) - z| < \delta$ for $t \in [0,1]$ and therefore

$$\begin{aligned} \Big|\frac{h(\xi) - h(z)}{\xi - z} - f(z)\Big| &= \Big|\int_{[\gamma]} \frac{f(\zeta) - f(z)}{z - \xi}\, d\zeta\Big| = \Big|\int_0^1 \frac{f(\gamma(t)) - f(z)}{z - \xi}\gamma'(t)\, dt\Big| \\ &\le \int_0^1 \big|f(\gamma(t)) - f(z)\big|\, dt < \int_0^1 \epsilon\, dt = \epsilon. \end{aligned}$$

Consequently $h'(z) = f(z)$ for every $|z| < R$.

11. [Carrier, Krook and Pearson Section 2-2 Exercise 8c] Show that a power series converges to an analytic function within its circle of convergence.

Consider the power series for $f(z)$ defined above with radius of convergence equal to R. By part a the function $g(z)$ also has radius of convergence equal to R. Since $f(z)$ may be obtained from $g(z)$ through term by term integration of $g(z)$ we have by part b that $f'(z) = g(z)$ for every z such that $|z| < R$. Thus, f is differentiable and therefore analytic in its circle of convergence.