1. [Walnut Exercise 4.44] Prove that if $\hat{x}_n$ is the N-point discrete Fourier transform of the period-N signal x_j then

$$\sum_{j=0}^{N-1} |x_j|^2 = \frac{1}{N}\sum_{n=0}^{N-1} |\hat{x}_n|^2.$$

By definition

$$\hat{x}_n = \sum_{j=0}^{N-1} x_j e^{-2\pi i j n/N}.$$

Therefore

$$\sum_{n=0}^{N-1} |\hat{x}_n|^2 = \sum_{n=0}^{N-1} \hat{x}_n \overline{\hat{x}_n} = \sum_{n=0}^{N-1} \Big(\sum_{j=0}^{N-1} x_j e^{-2\pi i j n/N}\Big)\overline{\Big(\sum_{k=0}^{N-1} x_k e^{-2\pi i k n/N}\Big)}$$

$$= \sum_{n=0}^{N-1}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1} x_j \overline{x_k} e^{-2\pi i (j-k)n/N} = \sum_{j=0}^{N-1}\sum_{k=0}^{N-1} x_j \overline{x_k} \sum_{n=0}^{N-1} e^{-2\pi i (j-k)n/N}.$$

Claim

$$\sum_{n=0}^{N-1} e^{-2\pi i (j-k)n/N} = \begin{cases} N & \text{if } j=k \\ 0 & \text{otherwise.}\end{cases}$$

Suppose $j = k$. Then $e^{-2\pi i (j-k)/N} = 1$. In this case it follows that

$$\sum_{n=0}^{N-1} e^{-2\pi i (j-k)n/N} = \sum_{n=0}^{N-1} 1 = N.$$

Now suppose $j \neq k$. Since j and k are between 0 and $N-1$ it follows that $j-k$ is not divisible by N. Therefore $e^{-2\pi i (j-k)/N} \neq 1$ and we have

$$\sum_{n=0}^{N-1} e^{-2\pi i (j-k)n/N} = \frac{1 - e^{-2\pi i (j-k)N/N}}{1 - e^{-2\pi i (j-k)/N}} = \frac{1-1}{1 - e^{-2\pi i (j-k)/N}} = 0.$$

In light of the claim, the only non-zero terms in the sum are when $j = k$. Therefore

$$\sum_{n=0}^{N-1} |\hat{x}_n|^2 = \sum_{j=0}^{N-1} x_j \overline{x_j} N = N \sum_{j=0}^{N-1} |x_j|^2,$$

which was to be shown.

Note the problem as originally stated in Walnut is missing a factor of N. The presence of the factor N and the correctness of the above derivation may be verified numerically using the Matlab/Octave code

```
x=rand(1024,1);
xhat=fft(x);
norm(xhat,2)^2/norm(x,2)^2
```

which gives the output

```
ans =  1024
```

showing that $\sum |\hat{x}_j|^2$ is indeed $N = 1024$ times larger than $\sum |x_n|^2$.

2. [Handout Problem 1] Compute the 36 entries of the matrix $\mathbf{W}_6$ corresponding to the discrete Fourier transform $\hat{x} = \mathbf{W}_6 x$ where $\hat{x}$ and x are vectors of length 6 given by

$$\hat{x} = \begin{bmatrix} \hat{x}_0 \\ \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}^5 \end{bmatrix}, \qquad x = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x^5 \end{bmatrix} \qquad \text{and} \qquad \hat{x}_n = \sum_{j=0}^{5} x_j e^{-2\pi i j n/6}.$$

The matrix is given by

$$[\mathbf{W}_6]_{j,k} = \omega^{(j-1)(k-1)} \qquad \text{where} \qquad \omega = e^{-\pi i/3}.$$

Since $\omega^3 = -1$ we can write $\mathbf{W}_6$ as

$$\mathbf{W}_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & -1 & -\omega & -\omega^2 \\ 1 & \omega^2 & -\omega & 1 & \omega^2 & -\omega \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\omega & \omega^2 & 1 & -\omega & \omega^2 \\ 1 & -\omega^2 & -\omega & -1 & \omega^2 & \omega \end{pmatrix}.$$

Using the fact that $\omega = \frac{1}{2} - i\frac{\sqrt{3}}{2}$ we obtain

$$\mathbf{W}_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{2} - \frac{\sqrt{3}\,i}{2} & -\frac{\sqrt{3}\,i}{2} - \frac{1}{2} & -1 & \frac{\sqrt{3}\,i}{2} - \frac{1}{2} & \frac{\sqrt{3}\,i}{2} + \frac{1}{2} \\ 1 & -\frac{\sqrt{3}\,i}{2} - \frac{1}{2} & \frac{\sqrt{3}\,i}{2} - \frac{1}{2} & 1 & -\frac{\sqrt{3}\,i}{2} - \frac{1}{2} & \frac{\sqrt{3}\,i}{2} - \frac{1}{2} \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \frac{\sqrt{3}\,i}{2} - \frac{1}{2} & -\frac{\sqrt{3}\,i}{2} - \frac{1}{2} & 1 & \frac{\sqrt{3}\,i}{2} - \frac{1}{2} & -\frac{\sqrt{3}\,i}{2} - \frac{1}{2} \\ 1 & \frac{\sqrt{3}\,i}{2} + \frac{1}{2} & \frac{\sqrt{3}\,i}{2} - \frac{1}{2} & -1 & -\frac{\sqrt{3}\,i}{2} - \frac{1}{2} & \frac{1}{2} - \frac{\sqrt{3}\,i}{2} \end{pmatrix}.$$

Note the above output was generated by the Macsyma/Maxima script

```
kill(all);
texout(var,file):=block (
    system(concat("rm -f ",file)),
    tex(var,file));
h[j,k]:=omega^((j-1)*(k-1));
w6:genmatrix(h,6,6);
let(omega^3,-1);
w6s:letsimp(w6);
texout(w6s,"hw2p2a.tex");
w6r:expand(ratsimp(subst(omega=exp(-%i*%pi/3),w6)));
texout(w6r,"hw2p2b.tex");
```

to avoid typographic errors.

3. [Handout Problem 2] Show that $\mathbf{W}_6$ can be factored as

$$\mathbf{W}_6 = \begin{bmatrix} I_3 & \Omega_3 \\ I_3 & -\Omega_3 \end{bmatrix} \begin{bmatrix} \mathbf{W}_3 & 0 \\ 0 & \mathbf{W}_3 \end{bmatrix} P_6$$

where I_3 is the identity matrix, Ω_3 is diagonal, $\mathbf{W}_3$ is the matrix corresponding to the discrete Fourier transofrm of length 3 and P_6 is a permutation matrix. Explicitly write out $\mathbf{W}_3$, Ω_3 and P_6.

By definition $\mathbf{W}_3$ is given by

$$\mathbf{W}_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 \\ 1 & \zeta^2 & \zeta \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & -\omega \\ 1 & -\omega & \omega^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -\frac{\sqrt{3}\,i}{2} - \frac{1}{2} & \frac{\sqrt{3}\,i}{2} - \frac{1}{2} \\ 1 & \frac{\sqrt{3}\,i}{2} - \frac{1}{2} & -\frac{\sqrt{3}\,i}{2} - \frac{1}{2} \end{pmatrix}$$

where $\zeta = e^{-2\pi i/3} = \omega^2$. Now, following Walnut equation (4.10) we have that Ω_3 is the diagonal matrix with entries 1, ω and ω^2 given by

$$\Omega_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} - \frac{\sqrt{3}\,i}{2} & 0 \\ 0 & 0 & -\frac{\sqrt{3}\,i}{2} - \frac{1}{2} \end{pmatrix}$$

and

$$P_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Multiplying these matrices in accordance with the suggested factorization yields

$$\begin{bmatrix} I_3 & \Omega_3 \\ I_3 & -\Omega_3 \end{bmatrix} \begin{bmatrix} \mathbf{W}_3 & 0 \\ 0 & \mathbf{W}_3 \end{bmatrix} P_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & -1 & -\omega & -\omega^2 \\ 1 & \omega^2 & -\omega & 1 & \omega^2 & -\omega \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\omega & \omega^2 & 1 & -\omega & \omega^2 \\ 1 & -\omega^2 & -\omega & -1 & \omega^2 & \omega \end{pmatrix}$$

which is the same as $\mathbf{W}_6$ as required.

All matrices appearing above and the verification of the factorization has been checked with a computer algebra system to avoid computational errors. In particular, Macsyma/Maxima was run with the script

```
kill(all);
texout(var,file):=block (
    system(concat("rm -f ",file)),
    tex(var,file));
h[j,k]:=zeta^((j-1)*(k-1));
w3:genmatrix(h,3,3);
let(zeta^3,1);
let(omega^3,-1);
orexp:exp(-%i*%pi/3);
w3s:letsimp(w3);
w3r:letsimp(subst(zeta=omega^2,w3));
w3q:expand(ratsimp(subst(omega=orexp,w3r)));
omega3:zeromatrix(3,3);
omega3[1,1]:1;
omega3[2,2]:omega;
omega3[3,3]:omega^2;
omega3r:expand(ratsimp(subst(omega=orexp,omega3)));
p6:matrix([1,0,0,0,0,0],[0,0,1,0,0,0],[0,0,0,0,1,0],
    [0,1,0,0,0,0],[0,0,0,1,0,0],[0,0,0,0,0,1]);
I3:ident(3);
z3:zeromatrix(3,3);
m1:addrow(addcol(I3,omega3),addcol(I3,-omega3));
m2:addrow(addcol(w3r,z3),addcol(z3,w3r));
w6t:letsimp(m1.m2.p6);
texout(w3s,"hw2p3a.tex");
texout(w3r,"hw2p3b.tex");
texout(omega3,"hw2p3c.tex");
texout(p6,"hw2p3d.tex");
texout(w6t,"hw2p3e.tex");
texout(w3q,"hw2p3f.tex");
texout(omega3r,"hw2p3g.tex");
```

to create the files `hw2p3a.tex`, `hw2p3b.tex`, `hw2p3c.tex`, `hw2p3d.tex`, and `hw3p3e.tex` that were included on the previous page of this document.

4. [Handout Problem 3] Show that $\mathbf{W}_6$ can be factored as

$$\mathbf{W}_6 = \begin{bmatrix} I_2 & \mathrm{X}_2 & \Psi_2 \\ I_2 & c\mathrm{X}_2 & c^2\Psi_2 \\ I_2 & c^2\mathrm{X}_2 & c\Psi_2 \end{bmatrix} \begin{bmatrix} \mathbf{W}_2 & 0 & 0 \\ 0 & \mathbf{W}_2 & 0 \\ 0 & 0 & \mathbf{W}_2 \end{bmatrix} Q_6$$

where I_2 is the identity matrix, X_2 and Ψ_2 are diagonal, c is a complex constant, $\mathbf{W}_2$ is the matrix corresponding to the discrete Fourier transform of length 2 and Q_6 is a permutation matrix. Explicitly write out $\mathbf{W}_2$, X_2, Ψ_2, c and Q_6.

Following the pattern for P_6, the permutation matrix Q_6 that breaks W_6 into three transforms of length two is

$$Q_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{W}_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Now

$$\begin{bmatrix} \mathbf{W}_2 & 0 & 0 \\ 0 & \mathbf{W}_2 & 0 \\ 0 & 0 & \mathbf{W}_2 \end{bmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

and so

$$\mathbf{W}_6 Q_6^{-1} \begin{bmatrix} \mathbf{W}_2 & 0 & 0 \\ 0 & \mathbf{W}_2 & 0 \\ 0 & 0 & \mathbf{W}_2 \end{bmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \omega & 0 & \omega^2 \\ 1 & 0 & \omega^2 & 0 & -\omega & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 0 & -\omega & 0 & \omega^2 & 0 \\ 0 & 1 & 0 & -\omega^2 & 0 & -\omega \end{pmatrix}.$$

From here it is easy to identify

$$\mathrm{X}_2 = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}, \qquad \Psi_2 = \begin{pmatrix} 1 & 0 \\ 0 & \omega^2 \end{pmatrix} \qquad \text{and} \qquad c = \omega^2.$$

Therefore,

$$\mathrm{X}_2 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} - \frac{\sqrt{3}\,i}{2} \end{pmatrix}, \qquad \Psi_2 = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{\sqrt{3}\,i}{2} - \frac{1}{2} \end{pmatrix} \qquad \text{and} \qquad c = -\frac{\sqrt{3}\,i}{2} - \frac{1}{2}.$$

Following is the Macsyma/Maxima script used for the above calculation:

```
kill(all);
texout(var,file):=block (
    system(concat("rm -f ",file)),
    tex(var,file));
q6:matrix([1,0,0,0,0,0],[0,0,0,1,0,0],
    [0,1,0,0,0,0],[0,0,0,0,1,0],
    [0,0,1,0,0,0],[0,0,0,0,0,1]);
w2:matrix([1,1],[1,-1]);
z2:zeromatrix(2,2);
h[j,k]:=omega^((j-1)*(k-1));
orexp:exp(-%pi*%i/3);
w6:genmatrix(h,6,6);
let(omega^3,-1);
w6s:letsimp(w6);
d1:addrow(addcol(w2,z2,z2),addcol(z2,w2,z2),addcol(z2,z2,w2));
d1inv:invert(d1);
t2:w6s.invert(q6).d1inv;
x2:submatrix(3,4,5,6,t2,1,2,5,6);
phi2:submatrix(3,4,5,6,t2,1,2,3,4);
x2r:expand(ratsimp(subst(omega=orexp,x2)));
phi2r:expand(ratsimp(subst(omega=orexp,phi2)));
texout(q6,"hw2p4a.tex");
texout(d1inv,"hw2p4b.tex");
texout(t2,"hw2p4c.tex");
texout(x2,"hw2p4d.tex");
texout(phi2,"hw2p4e.tex");
texout(x2r,"hw2p4f.tex");
texout(phi2r,"hw2p4g.tex");
texout(expand(ratsimp(orexp^2)),"hw2p4h.tex");
```

5. [Carrier, Krook and Pearson Section Section 2-3 Exercise 2] If $\Phi(z)$ is analytic in a simply connected region in which a closed contour C is drawn, obtain all possible values of

$$\int_C \frac{\Phi(\zeta)}{\zeta^2 - z^2}\, d\zeta$$

where both z and $-z$ are not points on C itself.

Case $z \neq 0$. Then the partial fractions decomposition

$$\frac{1}{\zeta^2 - z^2} = \frac{1}{2z(\zeta - z)} - \frac{1}{2z(\zeta + z)}$$

implies

$$\int_C \frac{\Phi(\zeta)}{\zeta^2 - z^2}\, d\zeta = \frac{1}{2z}\int_C \frac{\Phi(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2z}\int_C \frac{\Phi(\zeta)}{\zeta + z}\, d\zeta.$$

By Cauchy's Integral Formula

$$\frac{1}{2z}\int_C \frac{\Phi(\zeta)}{\zeta - z}\, d\zeta = \begin{cases} \dfrac{\pi i \Phi(z)}{z} & \text{if } z \text{ is inside the contour } C \\ 0 & \text{if } z \text{ is not inside the contour } C \end{cases}$$

and

$$\frac{1}{2z}\int_C \frac{\Phi(\zeta)}{\zeta + z}\, d\zeta = \begin{cases} \dfrac{\pi i \Phi(-z)}{z} & \text{if } -z \text{ is inside the contour } C \\ 0 & \text{if } -z \text{ is not inside the contour } C. \end{cases}$$

Therefore there are four possible values for the integral.

$$\int_C \frac{\Phi(\zeta)}{\zeta^2 - z^2}\, d\zeta = \begin{cases} \pi i\big(\Phi(z) - \Phi(-z)\big)/z & \text{if both } z \text{ and } -z \text{ are inside } C \\ \pi i \Phi(z)/z & \text{if only } z \text{ is inside } C \\ -\pi i \Phi(-z)/z & \text{if only } -z \text{ is inside } C \\ 0 & \text{if neither } z \text{ nor } -z \text{ are inside } C. \end{cases}$$

Case $z = 0$. Then

$$\int_C \frac{\Phi(\zeta)}{\zeta^2 - z^2}\, d\zeta = \int_C \frac{\Phi(\zeta)}{\zeta^2}\, d\zeta = \begin{cases} 2\pi i\, \Phi'(0) & \text{if } 0 \text{ is inside the contour } C \\ 0 & \text{if } 0 \text{ is not inside the contour } C. \end{cases}$$

Note that the original problem stated in Carrier, Krook and Pearson did not assume $-z$ was not on C. If $-z$ were on C then we must compute the principle value of the integral. In particular

$$\frac{1}{2z}\mathrm{PV}\!\!\int_C \frac{\Phi(\zeta)}{\zeta + z}\, d\zeta = \frac{\alpha}{2\pi}\frac{\pi i \Phi(-z)}{z} = \frac{\alpha i \Phi(-z)}{2z}$$

where $\alpha = \pi$ if C is smooth at $-z$ and otherwise α is some number between 0 and 2π depending on the angle α at the point $-z$ of the contour.

6. [Carrier, Krook and Pearson Section Section 2-3 Exercise 3] If n is an integer, positive or negative, and if C is a closed contour around the origin, use

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta \qquad \text{when } z \text{ is inside } C$$

to show that

$$\int_C \frac{d\zeta}{\zeta^n} = 0 \qquad \text{unless } n = 1.$$

Case $n = 1$. Taking $f(\zeta) = 1$ and observing that 0 is inside C we obtain

$$\int_C \frac{d\zeta}{\zeta} = 2\pi i \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - 0}\,d\zeta = 2\pi i f(0) = 2\pi i.$$

Case $n > 1$. Since $f(\zeta) = 1$ then $f^{(n-1)}(\zeta) = 0$. Therefore

$$\int_C \frac{d\zeta}{\zeta^n} = \frac{2\pi i}{(n-1)!}\frac{(n-1)!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - 0)^n}\,d\zeta = \frac{2\pi i}{(n-1)!} f^{(n-1)}(0) = 0.$$

Case $n < 0$. Then Cauchy's theorem implies

$$\int_C \frac{d\zeta}{\zeta^n} = \int_C \zeta^{|n|}\,d\zeta = 0$$

since $\zeta^{|n|}$ is a polynomial and therefore analytic on all of $\mathbf{C}$.

7. [Carrier, Krook and Pearson Section Section 2-3 Exercise 6] Given a continuous function $\Phi(z)$ and a closed contour C define

$$f(z) = \frac{1}{2\pi i}\int_C \frac{\Phi(\zeta)}{\zeta - z}\,d\zeta$$

for any point z inside C. Show $f(z)$ is analytic inside C. Show there exists a $\Phi(z)$, a contour C and a point z_0 on C such that $f(z)$ does not approach $\Phi(z_0)$ as $z \to z_0$.

To show $f(z)$ is analytic inside C we must show it is differentiable at each z inside C. We do this by using uniform convergence to pass a limit through the Cauchy integral formula. Given a point z inside C. Define $a = \min\{\,|\zeta - z| : \zeta \in C\,\}$. Note that $a > 0$ since C is a compact set. Claim

$$\frac{\xi - z}{\zeta - \xi} \to 0 \quad \text{uniformly for} \quad \zeta \in C \quad \text{as} \quad \xi \to z.$$

Let $\epsilon > 0$. Choose $\delta = \min(\epsilon a/2, a/2)$. Then $0 < |\xi - z| < \delta$ implies

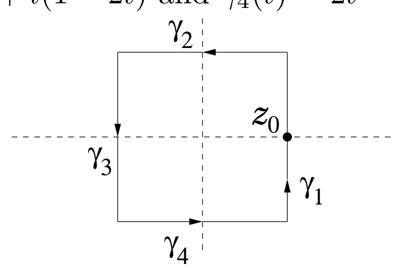
$$\left|\frac{\xi - z}{\zeta - \xi}\right| \le \frac{|\xi - z|}{|\zeta - z| - |z - \xi|} < \frac{\delta}{a - \delta} \le \frac{\epsilon a/2}{a - a/2} = \epsilon$$

thereby proving the claim. Now, the limit

$$\begin{aligned}
\lim_{\xi\to z}\frac{f(\xi) - f(z)}{\xi - z} &= \lim_{\xi\to z}\frac{1}{\xi - z}\left(\frac{1}{2\pi i}\int_C \frac{\Phi(\zeta)}{\zeta - \xi}\,d\zeta - \frac{1}{2\pi i}\int_C \frac{\Phi(\zeta)}{\zeta - z}\,d\zeta\right)\\
&= \frac{1}{2\pi i}\lim_{\xi\to z}\int_C \frac{\Phi(\zeta)}{(\zeta - \xi)(\zeta - z)}\,d\zeta\\
&= \frac{1}{2\pi i}\lim_{\xi\to z}\int_C \frac{\Phi(\zeta)}{(\zeta - z)^2}\,\frac{\zeta - z}{\zeta - \xi}\,d\zeta\\
&= \frac{1}{2\pi i}\lim_{\xi\to z}\int_C \frac{\Phi(\zeta)}{(\zeta - z)^2}\left(1 + \frac{\xi - z}{\zeta - \xi}\right)d\zeta\\
&= \frac{1}{2\pi i}\int_C \frac{\Phi(\zeta)}{(\zeta - z)^2}\,d\zeta
\end{aligned}$$

shows that the derivative $f'(z)$ exists. Therefore $f(z)$ is analytic inside C.

We now construct an example showing that $f(z)$ may not converge to $\Phi(z_0)$ as $z \to z_0$ where z_0 is on C and the z are points inside C. Since for any analytic function $\Phi(z)$ we have $f(z) = \Phi(z) \to \Phi(z_0)$ as $z \to z_0$, then we must take $\Phi(z)$ to be non-analytic. Let $\Phi(z) = \mathbf{Re}(z) = x$ where $z = x + iy$, let $C = [\gamma_1] + [\gamma_2] + [\gamma_3] + [\gamma_4]$ where $\gamma_1(t) = 1 + i(2t - 1)$, $\gamma_2(t) = 1 - 2t + i$, $\gamma_3(t) = -1 + i(1 - 2t)$ and $\gamma_4(t) = 2t - 1 - i$, and let $z_0 = 1$.



Now

$$I_1(x) = \int_{[\gamma_1]} \frac{\Phi(\zeta)}{\zeta - x}\, d\zeta = -2\,i \int_0^1 \frac{1}{x - 2\,i\,t + i - 1}\; dt$$
$$= \int_0^1 \frac{2\,i\;(1-x)}{(1-x)^2 + (2\,t-1)^2} + \frac{2\;(2\,t-1)}{(1-x)^2 + (2\,t-1)^2}\; dt,$$

$$I_2(x) = \int_{[\gamma_2]} \frac{\Phi(\zeta)}{\zeta - x}\, d\zeta = -\int_0^1 \frac{4\,t-2}{x + 2\,t - i - 1}\; dt$$
$$= \int_0^1 \frac{2\,i\;(1-2\,t)}{(-x-2\,t+1)^2+1} - \frac{2\;(1-2\,t)\;(-x-2\,t+1)}{(-x-2\,t+1)^2+1}\; dt,$$

$$I_3(x) = \int_{[\gamma_3]} \frac{\Phi(\zeta)}{\zeta - x}\, d\zeta = -2\,i \int_0^1 \frac{1}{x + 2\,i\,t - i + 1}\; dt$$
$$= \int_0^1 \frac{2\,i\;(-x-1)}{(-x-1)^2 + (1-2\,t)^2} + \frac{2\;(1-2\,t)}{(-x-1)^2 + (1-2\,t)^2}\; dt$$

and

$$I_4(x) = \int_{[\gamma_4]} \frac{\Phi(\zeta)}{\zeta - x}\, d\zeta = -\int_0^1 \frac{4\,t-2}{x - 2\,t + i + 1}\; dt$$
$$= \int_0^1 \frac{2\;(2\,t-1)\;(-x+2\,t-1)}{(-x+2\,t-1)^2+1} + \frac{2\,i\;(2\,t-1)}{(-x+2\,t-1)^2+1}\; dt$$

imply

$$\lim_{x\to 1^-} I_1(x) = \lim_{x\to 1^-} -2\,i\;\arctan\left(\frac{1}{x-1}\right) = i\,\pi,$$

$$\lim_{x\to 1^-} I_2(x) = -\frac{i\;\log 5}{2} + \frac{\log 5}{2} + i\;\arctan 2 + \arctan 2 - 2,$$

$$\lim_{x\to 1^-} I_3(x) = \lim_{x\to 1^-} -2\,i\;\arctan\left(\frac{1}{x+1}\right) = -2\,i\;\arctan\left(\frac{1}{2}\right)$$

and

$$\lim_{x\to 1^-} I_4(x) = -\frac{i\;\log 5}{2} - \frac{\log 5}{2} + i\;\arctan 2 - \arctan 2 + 2.$$

Consequently

$$\lim_{x\to 1^-} f(x) = \frac{1}{2\pi i} \lim_{x\to 1^-} \big(I_1(x) + I_2(x) + I_3(x) + I_4(x)\big)$$
$$= -\frac{\log 5}{2\,\pi} + \frac{\arctan 2}{\pi} - \frac{\arctan\left(\frac{1}{2}\right)}{\pi} + \frac{1}{2}$$
$$\approx 0.44868276533575.$$

However

$$\Phi(1) = \mathbf{Re}(1) = 1.$$

Therefore $f(x)$ does not tend to $\Phi(1)$ as $x \to 1$.

The calculations on the previous page were performed using Macsyma/Maxima.

```
kill(all);
texout(var,file):=block (
    system(concat("rm -f ",file)),
    tex(var,file));
g[1](t):=1+%i*(2*t-1);
g[2](t):=1-2*t+%i;
g[3](t):=-1+%i*(1-2*t);
g[4](t):=2*t-1-%i;
B(g):=ratsimp(realpart(g(t))*diff(g(t),t)/(g(t)-x));
I(g):=rectform(realpart(g(t))*diff(g(t),t)/(g(t)-x));
for i:1 thru 4 do (
    t0:'integrate(B(g[i]),t,0,1),
    texout(t0,concat("hw2p7b",string(i),".tex")),
    t1:'integrate(I(g[i]),t,0,1),
    texout(t1,concat("hw2p7a",string(i),".tex")),
    t2:ev(t1,nouns),
    J[i]:expand(ratsimp(ev(t2,nouns))),
    texout(J[i],concat("hw2p7c",string(i),".tex")),
    K[i]:expand(ratsimp(limit(J[i],x,1,minus))),
    texout(K[i],concat("hw2p7d",string(i),".tex")));
nonzero;
nonzero;
T1:sum(K[i],i,1,4);
T2:expand(ratsimp(T1/(2*%pi*%i)));
texout(T2,"hw2p7e.tex");
T3:float(T2);
texout(T3,"hw2p7f.tex");
```

8. [Carrier, Krook and Pearson Section Section 2-4 Exercise 1] Find the maximum for $|z| \leq 1$ of $|z^2+2z+i|$, $|\sin z|$ and $|\arcsin(z/2)|$.

By the maximum modulus theorem the maximum will be on the boundary $|z|=1$.

Part 1: Let $f(z) = z^2 + 2\,z + i$ and $z = e^{it}$. Then

$$f(z)\overline{f(z)} = f(e^{it})\overline{f(e^{it})} = \left(2\,e^{-i\,t} + e^{-2\,i\,t} - i\right)\,\left(e^{2\,i\,t} + 2\,e^{i\,t} + i\right).$$

Setting the derivative with respect to t equal to zero yields

$$\frac{d}{dt}\left(f(e^{it})\overline{f(e^{it})}\right) = e^{-2\,i\,t}\,\left(2\,e^{4\,i\,t} + (2\,i+2)\,e^{3\,i\,t} + (2-2\,i)\,e^{i\,t} + 2\right) = 0.$$

Therefore

$$\frac{2\,z^4 + (2\,i+2)\,z^3 + (2-2\,i)\,z + 2}{z^2} = \frac{2\,(z+1)\,(z+i)\,\left(z^2-i\right)}{z^2} = 0$$

and the critical points are

$$z_1 = -\left(-1\right)^{\frac{1}{4}}, \quad z_2 = \left(-1\right)^{\frac{1}{4}}, \quad z_3 = -i \quad \text{and} \quad z_4 = -1.$$

Plugging these critical points into $|f(z)|$ yields

$$\begin{aligned}
|f(z_1)| &= \sqrt{8-2^{\frac{5}{2}}} \approx 1.530733729460358\\
|f(z_2)| &= \sqrt{2^{\frac{5}{2}}+8} \approx 3.695518130045147\\
|f(z_3)| &= \sqrt{2} \approx 1.414213562373095\\
|f(z_4)| &= \sqrt{2} \approx 1.414213562373095.
\end{aligned}$$

Therefore the maximum is $|f(z_2)| = \sqrt{2^{\frac{5}{2}}+8} \approx 3.695518130045147$.

Calculations were done with Macsyma/Maxima.

```
kill(all);
texout(var,file):=block (
    system(concat("rm -f ",file)),
    tex(var,file));
f:z^2+2*z+%i;
texout(f,"hw2p8a1.tex");
fe:subst(z=exp(%i*t),f);
ff:fe*conjugate(fe);
texout(ff,"hw2p8a2.tex");
dff:ratsimp(diff(ff,t));
texout(dff,"hw2p8a3.tex");
dffz:subst(t=log(z)/%i,dff);
```

```
texout(dffz,"hw2p8a4.tex");
dffz2:gfactor(dffz);
texout(dffz2,"hw2p8a5.tex");
cp:solve(dffz2=0,z);
mi:0;
mv:0;
mr:0;
for i:1 thru 4 do (
    fz:subst(cp[i],f),
    texout(subst(cp[i],z),concat("hw2p8a6-",string(i),".tex")),
    absfz:expand(abs(fz)),
    texout(absfz,concat("hw2p8a7-",string(i),".tex")),
    absfzr:float(absfz),
    texout(absfzr,concat("hw2p8a8-",string(i),".tex")),
    if mr<absfzr then (
        mi:i,
        mv:absfz,
        mr:absfzr));
texout(mi,"hw2p8a9-1.tex");
texout(mv,"hw2p8a9-2.tex");
texout(mr,"hw2p8a9-3.tex");
```

Part 2: Let $f(z) = \sin z$ and $z = e^{it}$. Then

$$f(z)\overline{f(z)} = f(e^{it})\overline{f(e^{it})} = \sin e^{-i\,t}\, \sin e^{i\,t}.$$

Setting the derivative with respect to t equal to zero yields

$$\frac{d}{dt}\Big(f(e^{it})\overline{f(e^{it})}\Big) = -e^{-i\,t}\,\left(i\,\cos e^{-i\,t}\,\sin e^{i\,t} - i\,e^{2\,i\,t}\,\sin e^{-i\,t}\,\cos e^{i\,t}\right) = 0.$$

Therefore

$$-\frac{i\,\cos\left(\frac{1}{z}\right)\,\sin z - i\,\sin\left(\frac{1}{z}\right)\,z^2\,\cos z}{z} = 0$$

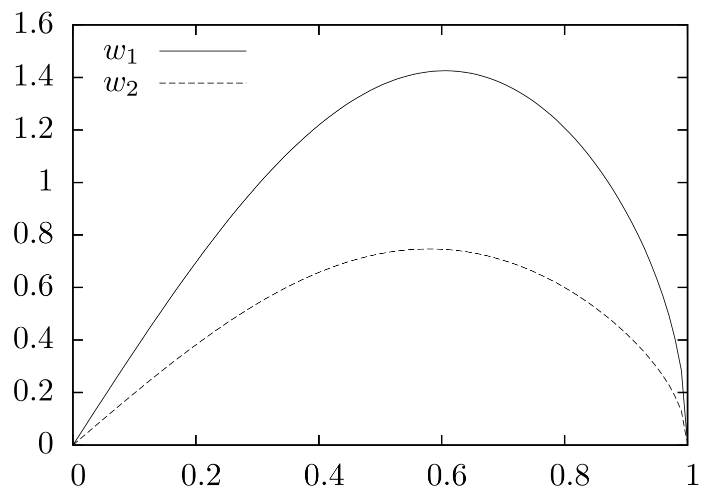
Since $|\sin z| = |\sin\overline{z}|$ we can assume the imaginary part of z is positive. Since $|z| = 1$ then we can write $z = x + i\sqrt{1-x^2}$ where $|x| \le 1$. Simplifying yields

$$2\,x\,\cosh\sqrt{1-x^2}\,\sinh\sqrt{1-x^2} - 2\,\sqrt{1-x^2}\,\cos x\,\sin x = 0.$$

By symmetry we can assume $x \ge 0$. Claim that $x_1 = 0$ and $x_2 = 1$ are the only solutions of the above equality for $0 \le x \le 1$. This can be seen graphically by plotting the functions

$$w_1(x) = 2\,x\,\cosh\sqrt{1-x^2}\,\sinh\sqrt{1-x^2} \qquad \text{and} \qquad w_2(x) = 2\,\sqrt{1-x^2}\,\cos x\,\sin x$$

and observing that the only place the two graphs intersect is at x_1 and x_2.



It follows that the critical points are

$$z_1 = 1, \quad z_2 = -1, \quad z_3 = i \quad \text{and} \quad z_4 = -i.$$

Plugging these critical points into $|f(z)|$ yields

$$|f(z_1)| = |f(z_2)| = |\sin 1| \approx 0.8414709848079$$

and

$$|f(z_3)| = |f(z_4)| = |\sin i| \approx 1.175201193643801.$$

Therefore the maximimum is $|\sin i| \approx 1.175201193643801$.

Calculations were performed with Macsyma/Maxima using the script

```
kill(all);
texout(var,file):=block (
    system(concat("rm -f ",file)),
    tex(var,file));
f:sin(z);
texout(f,"hw2p8b1.tex");
fe:subst(z=exp(%i*t),f);
ff:fe*conjugate(fe);
texout(ff,"hw2p8b2.tex");
dff:ratsimp(diff(ff,t));
texout(dff,"hw2p8b3.tex");
dffz:subst(t=log(z)/%i,dff);
texout(dffz,"hw2p8b4.tex");
t1:subst(z=x+%i*sqrt(1-x^2),dffz);
assume(abs(x)<1);
dffx:trigsimp(rectform(t1));
texout(dffx,"hw2p8b5.tex");
w1r:part(dffx,1);
w2r:-part(dffx,2);
texout(w1r,"hw2p8b6.tex");
texout(w2r,"hw2p8b7.tex");
with_stdout("hw2p8bw.f",
    fortran(w1(x)=w1r),
    fortran(w2(x)=w2r));
sr1:float(abs(sin(1)));
texout(sr1,"hw2p8b8.tex");
sr2:float(abs(sin(%i)));
texout(sr2,"hw2p8b9.tex");
```

and the plot was created with Gnuplot using the script

```
set terminal pstex
set output "hw2p8bw.tex"
set key left width -2.5
set size 0.7,0.6
load "hw2p8bw.f"
plot [0:1] w1(x) ti "$w_1$" ,w2(x) ti "$w_2$"
```

Part 3: Let $f(z) = \arcsin\left(\frac{z}{2}\right)$. Let $f(z) = \alpha + i\beta$ so that $\sin(\alpha + i\beta) = z/2$. By symmetry we may assume $\beta > 0$. Since $|z|^2 = 1$ it follows that

$$\begin{aligned}\frac{1}{4} = \frac{|z|^2}{4} &= \sin(\alpha + i\beta)\sin(\alpha - i\beta) = \cos^2\alpha\,\sinh^2\beta + \sin^2\alpha\,\cosh^2\beta \\ &= \sin^2\alpha\,\left(\sinh^2\beta + 1\right) + \left(1 - \sin^2\alpha\right)\,\sinh^2\beta \\ &= \sinh^2\beta + \sin^2\alpha.\end{aligned}$$

Solve for β to obtain

$$\beta = \operatorname{asinh}\left(\frac{\sqrt{1 - 4\,\sin^2\alpha}}{2}\right).$$

Therefore

$$\left|\arcsin(z/2)\right|^2 = \alpha^2 + \beta^2 = \alpha^2 + \operatorname{asinh}^2\left(\frac{\sqrt{1 - 4\,\sin^2\alpha}}{2}\right).$$

Let $4\sin^2\alpha = \omega$ where $0 \le \omega \le 1$. Then

$$\alpha^2 = \arcsin^2\left(\frac{\sqrt{\omega}}{2}\right)$$

and

$$\alpha^2 + \beta^2 = \arcsin^2\left(\frac{\sqrt{\omega}}{2}\right) + \operatorname{asinh}^2\left(\frac{\sqrt{1-\omega}}{2}\right).$$

The maximum of $|f(z)|$ may now be found by maximizing $\alpha^2 + \beta^2$ as a function of ω. Since $\arcsin\xi$ is a convex function on the interval $[0, 1/2]$ and

$$\arcsin 0 = 0, \qquad \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6} \qquad \text{and} \qquad \frac{d}{d\theta}\arcsin\xi\Big|_{\xi=0} = 1$$

then

$$\xi \le \arcsin\xi \le \frac{\pi\xi}{3} \qquad \text{for} \qquad 0 \le \xi \le \frac{1}{2}.$$

Since $\operatorname{asinh}\xi$ is a concave function on the interval $[0, 1/2]$ and

$$\operatorname{asinh} 0 = 0, \qquad \operatorname{asinh}\left(\frac{1}{2}\right) = \log\left(\frac{1+\sqrt{5}}{2}\right) \qquad \text{and} \qquad \frac{d}{d\theta}\operatorname{asinh}\xi\Big|_{\xi=0} = 1$$

then

$$2\xi\log\left(\frac{1+\sqrt{5}}{2}\right) \le \operatorname{asinh}\xi \le \xi \qquad \text{for} \qquad 0 \le \xi \le \frac{1}{2}.$$

Differentiating we obtain

$$\frac{d(\alpha^2+\beta^2)}{dw} = \frac{\arcsin\left(\frac{\sqrt{\omega}}{2}\right)\,\sqrt{1-\omega}\,\sqrt{5-\omega} - \operatorname{asinh}\left(\frac{\sqrt{1-\omega}}{2}\right)\,\sqrt{4-\omega}\,\sqrt{\omega}}{\sqrt{1-\omega}\,\sqrt{4-\omega}\,\sqrt{5-\omega}\,\sqrt{\omega}}.$$

Claim that the numerator of the derivative is non-negative. Equivalently

$$\arcsin\left(\frac{\sqrt{\omega}}{2}\right)\sqrt{1-\omega}\sqrt{5-\omega} \geq \operatorname{asinh}\left(\frac{\sqrt{1-\omega}}{2}\right)\sqrt{4-\omega}\sqrt{\omega} \qquad \text{for} \qquad 0 \leq \omega \leq 1.$$

Now, by the inequalities just proven about $\arcsin\xi$ and $\operatorname{asinh}\xi$ we have

$$\arcsin\left(\frac{\sqrt{\omega}}{2}\right)\sqrt{1-\omega}\sqrt{5-\omega} \geq \frac{\sqrt{\omega}}{2}\sqrt{1-\omega}\sqrt{5-\omega}$$

and

$$\operatorname{asinh}\left(\frac{\sqrt{1-\omega}}{2}\right)\sqrt{4-\omega}\sqrt{\omega} \leq \frac{\sqrt{1-\omega}}{2}\sqrt{4-\omega}\sqrt{\omega}.$$

Since $\sqrt{5-\omega} > \sqrt{4-\omega}$ the numerator is non-negative. Thus $\alpha^2+\beta^2$ is a non-decreasing function of ω and the maximum of $|f(z)|$ occurs when $\omega = 1$. The maximum is

$$\sqrt{\arcsin^2\left(\frac{\sqrt{\omega}}{2}\right) + \operatorname{asinh}^2\left(\frac{\sqrt{1-\omega}}{2}\right)}\Bigg|_{\omega=1} = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}.$$

Some calculations were done with Macsyma/Maxima using the script

```
kill(all);
texout(var,file):=block (
    system(concat("rm -f ",file)),
    tex(var,file));
f:asin(z/2);
texout(f,"hw2p8c1.tex");
fi:sin(alpha+%i*beta);
fii:rectform(fi*conjugate(fi));
texout(fii,"hw2p8c2.tex");
fii1:subst(cos(alpha)^2=1-sin(alpha)^2,fii);
fii2:subst(cosh(beta)^2=sinh(beta)^2+1,fii1);
texout(fii2,"hw2p8c3.tex");
fii2r:ratsimp(fii2);
texout(fii2r,"hw2p8c4.tex");
b1:solve(fii2r=1/4,sinh(beta)^2);
b2:asinh(sqrt(subst(b1,sinh(beta)^2)));
texout(b2,"hw2p8c5.tex");
bb:b2^2;
texout(bb,"hw2p8c6.tex");
w1:4*sin(alpha)^2=omega;
a1:asin(sqrt(omega)/2)^2;
texout(a1,"hw2p8c7.tex");
bb2:subst(w1,bb);
h1:a1+bb2;
texout(h1,"hw2p8c8.tex");
h2:ratsimp(diff(h1,omega));
texout(h2,"hw2p8c9.tex");
```