Math 761 Midterm Version A

1. Fill in the missing blanks in following definitions from from *An Introduction to Wavelet Analysis* by David Walnut.

Definition 4.35. (Discrete Fourier Transform) *Given a period N signal x_n, the N-point discrete Fourier transofrm of x_n, denoted $\hat{x}_n$, is the period N sequence defined by*

$$\hat{x}_n = \boxed{} \,.$$

Definition 5.9. (Haar Scaling Fuctions) *Let*

$$p(x) = \boxed{} \,,$$

and for each $j, k \in \mathbf{Z}$ define

$$p_{j,k}(x) = \boxed{} \,.$$

Then the collection $\{ p_{j,k}(x) : j, k \in \mathbf{Z} \}$ is called the system of Haar scaling functions and for each $J \in Z$ the collection $\{ p_{J,k}(x) : k \in \mathbf{Z} \}$ is referred to as the system of scale J Haar scaling functions.

Definition 5.11. (Haar System) *Let*

$$h(x) = \boxed{} \,,$$

and for each $j, k \in \mathbf{Z}$ define

$$h_{j,k}(x) = \boxed{} \,.$$

Then the collection $\{ h_{j,k}(x) : j, k \in \mathbf{Z} \}$ is called the Haar system and for each $J \in Z$ the collection $\{ h_{J,k}(x) : k \in \mathbf{Z} \}$ is referred to as the system of scale J Haar functions.

2. Prove one of the following:

(i) Discrete Fourier Inversion Theorem. Given a period N sequence x_n with discrete Fourier transform $\hat{x}_n$. Then

$$x_j = \frac{1}{N}\sum_{n=0}^{N-1} \hat{x}_n e^{2\pi i n j/N} \qquad \text{for each} \qquad j \in \mathbf{Z}.$$

(ii) Discrete Convolution Theorem. Let x_n and y_n be period N signals. Then

$$\widehat{(x * y)}_n = \hat{x}_n \hat{y}_n \qquad \text{where} \qquad (x * y)_n = \sum_{k=0}^{N-1} x_k y_{n-k}.$$

3. Prove one of the following.

(i) Splitting Theorem. $\operatorname{span}\{\, p_{j+1,k} : k = 0, 1, \ldots, 2^{j+1} - 1 \,\} = \operatorname{span}\{\, p_{j,k} : k = 0, 1, \ldots, 2^j - 1 \,\} \cup \{\, h_{j,k} : k = 0, 1, \ldots, 2^j - 1 \,\}$.

(ii) Haar Coefficient Decay Rates. Let $f \in C^1([0,1])$. Then $\langle f, h_{j,k} \rangle = \mathcal{O}(2^{-3j/2})$ as $j \to \infty$.

4. Suppose f is analytic in a neighborhood containing the disk

$$A = \{\,\zeta : |\zeta - z_0| < R\,\}.$$

Prove one of the following.

(i) If $|f(z)|$ attains its maximum at z_0. Then

$$\frac{1}{\pi R^2}\int_A |f(\zeta)|\,dA = |f(z_0)|.$$

(ii) For every $z \in A$ the Taylor series converges and

$$f(z) = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!} f^{(n)}(z_0).$$

5. The Haar functions $h_{j,k}$ and scaling functions $p_{j,k}$ satisfy

(A) $h_{j,k} = 2^{-1/2}(p_{j+1,2k} - p_{j+1,2k+1})$.

(B) $h_{j,k} = 2^{j/2}(p_{j+1,2k} - p_{j+1,2k+1})$.

(C) $h_{j,k} = 2^{-1/2}(p_{j+1,2k} + p_{j+1,2k+1})$.

(D) $h_{j,k} = 2^{j/2}(p_{j+1,2k} + p_{j+1,2k+1})$.

(E) none of these.

6. Let $\gamma(t) = 2e^{2\pi i t}$ where $t \in [0,1]$. Use Cauchy's integral formula to evaluate the following integrals.

(i) $\displaystyle\int_{[\gamma]} \frac{\sinh(\zeta^2 - 1)}{\zeta - 1}\, d\zeta$

(ii) $\displaystyle\int_{[\gamma]} \frac{\sinh(\zeta^2 - 1)}{\zeta^2 - \zeta}\, d\zeta$