Math 761 Quiz 1 Version A

Formulas and Definitions

Approximate Identity. A collection of functions on $\mathbf{R}$ is an approximate identity on $\mathbf{R}$ if the following conditions hold.

(a) For every $\tau > 0$ holds $\displaystyle\int_{\mathbf{R}} K_\tau(x)\,dx = 1$.

(b) There exists $M > 0$ such that for every $\tau > 0$ holds $\displaystyle\int_{\mathbf{R}} \big|K_\tau(x)\big|\,dx \le M$.

(c) For every $0 < \delta < a$ holds $\displaystyle\lim_{\tau\to 0+} \int_{\delta<|x|<a} \big|K_\tau(x)\big|\,dx = 0$.

Binomial Theorem. Let n and k be integers. Then

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \qquad \text{where} \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Cauchy's Product. Suppose $\sum_{n=0}^\infty a_n$ and $\sum_{m=0}^\infty b_n$ are absolutely convergent. Then

$$\Big(\sum_{n=0}^{\infty} a_n\Big)\Big(\sum_{m=0}^{\infty} b_m\Big) = \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} a_{n-k} b_k\Big).$$

Analytic. A function $f(z)$ is said to have a derivative at the point z if

$$\lim_{\xi\to z} \frac{f(\xi) - f(z)}{\xi - z}$$

exists and has the same value for any mode of approach of ξ to z. If $f(z)$ has a derivative at z_0 and also at each point in some neighborhood of z_0, then $f(z)$ is analytic at z_0.

Stokes Theorem. In one dimension Stokes theorem is known as the Fundamental Theorem of Calculus

$$\int_a^b \Big(\frac{df}{dx}\Big)\,dx = f(b) - f(a)$$

and in two dimensions it is frequently called Green's Theorem

$$\int_{\partial\Omega} P dx + Q dy = \iint_{\Omega} \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,dx\,dy$$

where P and Q are continuously differentiable scalar fields and the boundary $\partial\Omega$ is assumed to be a piecewise smooth Jordan curve oriented in the counterclockwise direction.

Math 761 Quiz 1 Version A

1. Fill in the missing blanks in the statements of the following theorems from *An Introduction to Wavelet Analysis* by David Walnut.

Theorem 2.16. (Dirichlet) *Suppose that $f(x)$ has period $a > 0$ and is*

[ ] *on* $\mathbf{R}$.

Then the sequence of partial sums $S_N(x)$ of the Fourier series of $f(x)$ where

$$S_N(x) = \sum_{n=-N}^{N} c_n e^{2\pi i n x/a} \qquad \textit{and} \qquad c_n = \boxed{}$$

converge pointwise to the function $\tilde{f}(x)$, where

$$\tilde{f}(x) = \boxed{}.$$

Theorem 2.19. (Fejér) *Let $f(x)$ be a function with period $a >$ that is*

[ ] *on* $\mathbf{R}$,

and define for $N \in \mathbf{N}$ the function $\sigma_N(x)$ by

$$\sigma_N(x) = \boxed{},$$

where $S_k(x)$ is defined as in Theorem 2.16. Then $\sigma_N(x)$

[ ] *to $f(x)$ on* $\mathbf{R}$ *as $N \to \infty$.*

Math 761 Quiz 1 Version A

2. Define

$$(f * h)(x) = \int_{-\infty}^{\infty} f(x-t)h(t)\,dt$$

Prove one of the following.

(i) There is no integrable function h such that $(f * h)(a) = f(a)$ holds for every function $f(x)$ that is L^∞ on $\mathbf{R}$ and continuous at $x = a$.

(ii) Let $f(x)$ be L^∞ on $\mathbf{R}$ and continuous at the point $x = a$. Suppose K_τ is an approximate identity on $\mathbf{R}$. Then $(f * K_\tau)(a) \to f(a)$ as $\tau \to 0$.

Math 761 Quiz 1 Version A

3. Prove one of the following.

(i) Let $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$ and u and v are real functions. Suppose f is differentiable at the point $x_0 + iy_0$. Show that the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ hold at this point.

(ii) Let $f(x)$ be analytic over some region R of the complex plane and consider any closed simple piecewise smooth Jordan curve C which together with its interior is completely contained in R. Prove Cauchy's theorem $\int_C f(z)\,dz = 0$.