Math 761 Quiz 2 Version A — Key —

1. Let $\gamma(t) = e^{2\pi i t}$ where $t \in [0,1]$ Use Cauchy's integral formula to evaluate the following integrals:

(i) $\displaystyle\int_{[\gamma]} \frac{\sin\zeta}{2\zeta+1}\,d\zeta = \frac{1}{2}\int_{[\gamma]} \frac{\sin\zeta}{\zeta+\frac{1}{2}}\,d\zeta = \pi i \,\frac{1}{2\pi i}\int_{[\gamma]} \frac{\sin\zeta}{\zeta+\frac{1}{2}}\,d\zeta$

$= \pi i \, \sin(-\tfrac{1}{2}) = -\pi i \sin(\tfrac{1}{2})$

since $-\frac{1}{2}$ is inside the curve $[\gamma]$.

(ii) $\displaystyle\int_{[\gamma]} \frac{\sin\zeta}{\zeta+2}\,d\zeta$ since -2 is outside the curve $[\gamma]$

then $\frac{\sin\zeta}{\zeta+2}$ is analytic everywhere inside $[\gamma]$. Thus

by Cauchy's theorem

$$\int_{[\gamma]} \frac{\sin\zeta}{\zeta+2}\,d\zeta = 0.$$

(iii) $\displaystyle\int_{[\gamma]} \frac{\sin\zeta}{2\zeta^2+1}\,d\zeta = \frac{1}{2}\int_{[\gamma]} \frac{\sin\zeta}{\zeta^2+\frac{1}{2}}\,d\zeta = \pi i\,\frac{1}{2\pi i}\int_{[\gamma]} \frac{\sin\zeta}{(\zeta-\frac{i}{\sqrt{2}})(\zeta+\frac{i}{\sqrt{2}})}\,d\zeta.$

$\displaystyle = \pi i \frac{1}{2\pi i}\int_\gamma \frac{A\sin\zeta}{\zeta-\frac{i}{\sqrt{2}}}\,d\zeta + \int_\gamma \frac{B\sin\zeta}{\zeta+\frac{i}{\sqrt{2}}}\,d\zeta = \pi i\left(A\sin\tfrac{i}{\sqrt{2}} + B\sin(\tfrac{-i}{\sqrt{2}})\right)$

$= \pi i (A-B)\sin(\frac{i}{\sqrt{2}}) = -\pi(A-B)\sinh(1/\sqrt{2}) = \pi i\sqrt{2}\sinh(1/\sqrt{2}).$

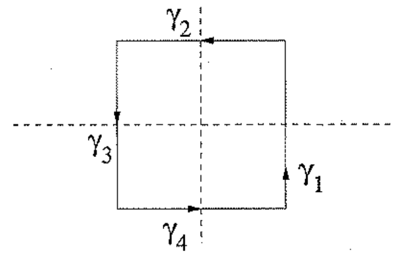
since the partial fraction decomposition

$A(\zeta+\frac{i}{\sqrt{2}}) + B(\zeta-\frac{i}{\sqrt{2}}) = 1, \quad A+B=0, \quad A\frac{i}{\sqrt{2}} - B\frac{i}{\sqrt{2}} = 1$

implies $A-B = \frac{\sqrt{2}}{i} = -i\sqrt{2}$.

Math 761 Quiz 2 Version A

2. Consider the curve given by $C = [\gamma_1] + [\gamma_2] + [\gamma_3] + [\gamma_4]$ where $\gamma_1(t) = 1 + i(2t-1)$, $\gamma_2(t) = 1 - 2t + i$, $\gamma_3(t) = -1 + i(1-2t)$ and $\gamma_4(t) = 2t - 1 - i$.



Find $\displaystyle\int_C \frac{i\cos(\zeta)}{\zeta^3 - 2\zeta^2}\, d\zeta. = \int_C \frac{i\cos\zeta}{\zeta^2(\zeta-2)}\, d\zeta = \frac{1}{2\pi i}\int_C \frac{-2\pi\cos\zeta}{\zeta^2(\zeta-2)}\, d\zeta$

$$= \frac{1}{2\pi i}\int \frac{f(\zeta)}{\zeta^2}\, d\zeta = f'(0) \text{ where } f(\zeta) = \frac{-2\pi\cos\zeta}{\zeta-2}.$$

By Cauchy's formula. Since

$$f'(\zeta) = \frac{-2\pi(\sin\zeta)(\zeta-2) + 2\pi\cos\zeta}{(\zeta-2)^2}$$

then

$$f'(0) = \frac{-2\pi(\sin 0)(-2) + 2\pi\cos 0}{(0-2)^2} = \frac{2\pi}{4} = \frac{\pi}{2}.$$

It follows that

$$\int_C \frac{i\cos\zeta}{\zeta^3 - 2\zeta^2}\, d\zeta = \frac{\pi}{2}$$

3. Prove if f is bounded on $[0,1]$ that $\langle f, h_{j,k}\rangle = O(2^{-j/2})$ as $j \to \infty$.

Let $B = \sup\{|f(t)| : t \in [0,1]\}$. Since f is bounded then B is finite. Now for $k = 0, \dots, 2^j - 1$ we have

$$|\langle f, h_{j,k}\rangle| = \left|\int_0^1 f(t) h_{j,k}(t)\, dt\right| \le \int_0^1 B\, |h_{j,k}(t)|\, dt$$

$$= \int_{k/2^j}^{(k+1)/2^j} B 2^{j/2}\, dt = B 2^{j/2}\left(\frac{k+1}{2^j} - \frac{k}{2^k}\right)$$

$$= B 2^{-j/2}.$$

Therefore

$$\langle f, h_{j,k}\rangle = O(2^{-j/2}) \text{ as } j \to \infty.$$

Math 761 Quiz 2 Version A

4. Prove one of the following:

(i) If $x_0 \in [0,1]$ is a dyadic irrational then there is an increasing sequence j_n of natural numbers such that

$$\frac{1}{4} \le 2^{j_n} x_0 \bmod 1 \le \frac{1}{2}.$$

(ii) Let $h_{j,k}$ be the functions which make up the Haar system. Define

$$a = \frac{k}{2^j}, \qquad b = \frac{k+1}{2^j} \qquad \text{and} \qquad c = \frac{a+b}{2}.$$

If $x_0 \in [a, c)$ then

$$\int_a^{x_0} 5h_{j,k}(t)\,dt + \int_{x_0}^b h_{j,k}(t)\,dt = 2^{2+j/2}(x_0 - a).$$

Part (i) is exactly the lemma for case 2 of the Haar coefficient decay rates.

Part (ii) is related to the first part of the proof of Theorem 2 concerning case 2 of the Haar coefficient decay rates.

In particular part (ii) is as follows:

Since $h_{j,k}(t) = \begin{cases} 2^{j/2} & \text{if } t \in [a,c) \\ -2^{j/2} & \text{if } t \in [c,b) \end{cases}$ then $x \in [a,c)$ implies

$$\int_a^{x_0} 5h_{j,k}(t)dt + \int_{x_0}^b h_{j,k}(t)dt = \int_a^{x_0} 5h_{j,k}(t)dt + \int_{x_0}^c h_{j,k}(t)\,dt + \int_c^b h_{j,k}(t)\,dt$$

$$= \int_a^{x_0} 5\,2^{j/2} dt + \int_{x_0}^c 2^{j/2} dt - \int_c^b 2^{j/2} dt = 2^{j/2}\big(5(x_0-a) + (c-x_0) - (b-c)\big)$$

$$= 2^{j/2}\big(5(x_0-a) + 2c - x_0 - b\big) = 2^{j/2}\big(5(x_0-a) + a + \cancel{b} - x_0 - \cancel{b}\big)$$

$$= 2^{j/2}\big(5(x_0-a) + (a-x_0)\big) = 2^{j/2} \cdot 4 \cdot (x_0-a) = 2^{2+\frac{j}{2}}(x_0-a),$$