August 31, 2011

Claim There is no function h such that

$$f * h = f$$

for every continuous function f.

Suppose such an h existed. Then taking $f(x)=1$ we would obtain that

$$1 = f(0) = f*h(0) = h*f(0) = \int_{\mathbb{R}} h(t) f(-t)\,dt = \int_{\mathbb{R}} h(t)\,dt$$

Choose $\delta > 0$ so that

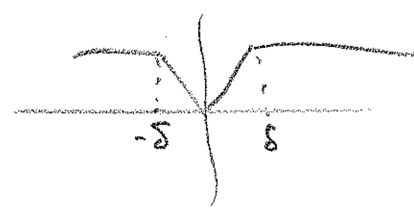
$$\int_{|t|\leq\delta} |h(t)|\,dt \leq \frac{1}{3}$$

Then

$$\int_{|t|>\delta} h(t)\,dt = 1 - \int_{|t|\leq\delta} h(t)\,dt >= 1 - \frac{1}{3} = \frac{2}{3}.$$

Define

$$f(x) = \begin{cases} \frac{|x|}{\delta} & \text{if } |x| \leq \delta \\ 1 & \text{otherwise} \end{cases}$$



Then f is continuous, symmetric and

$$0 = f(0) = \int_{\mathbb{R}} h(t) f(-t)\,dt = \int_{|t|>\delta} h(t)\,dt + \int_{|t|\leq\delta} h(t) f(t)\,dt$$

$$\geq \frac{2}{3} - \int_{|t|\leq\delta} |h(t)|\,dt = \frac{2}{3} - \frac{1}{3} = \frac{1}{3},$$

which is a contradiction.

Therefore, no such function h exists.

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Remark on why you can choose δ so that

$$\int_{|t|\le\delta} |h(t)|\,dt \le \frac{1}{3}$$

in the preceding proof.

Riemann Integral Way

For $\int_{\mathbb{R}} h(t)\,dt = \int_{-\infty}^{\infty} h(t)\,dt$ to make sense as an improper Riemann integral h needs to be bounded on any finite interval. Thus, there is B such that

$$|h(x)| \le B \quad \text{for } x \in [-1,1].$$

Now choosing $\delta = \min(1, \frac{1}{6B})$ we obtain that

$$\int_{|t|\le\delta} |h(t)|\,dt \le 2\delta B \le \frac{1}{3}$$

LEBESGUE Integral Way

For $\int_{\mathbb{R}} h(t)\,dt = 1$ to make sense as a Lebesgue integral we must have that $\int_{\mathbb{R}} |h(t)|\,dt < \infty$. It follows from the dominated convergence theorem that

$$\int_{|t|\le\frac{1}{n}} |h(t)|\,dt \to 0 \quad \text{as } n \to \infty.$$

The result then follows by taking $\delta = \frac{1}{n}$ for n suitably large.