Theorem. Suppose $F\colon \mathbf{R}\to\mathbf{C}$ is $L^1(\mathbf{R})$ integrable and piecewise continuous. Define $g_t(x) = f(x-t)$. Then

$$g_t \to f \qquad \text{in} \qquad L^1(\mathbf{R}) \qquad \text{as} \qquad t\to 0.$$

Riemann Integral Way: For simplicity assume f is everywhere continuous. Let $\epsilon > 0$ be arbitrary. Since f is $L^1(\mathbf{R})$ integrable then $\int_{-\infty}^{\infty} |f(x)|dx = L$ converges as an improper Riemann integral. Therefore, there is N large enough such that

$$\Big|\int_{-N}^{N} |f(x)|dx - L\Big| < \frac{\epsilon}{4} \qquad \text{or equivalently} \qquad \int_{|x|>N} |f(x)|dx < \frac{\epsilon}{4}.$$

Define $J = [-N-2, N+2]$. Since J is a closed bounded interval, then f is uniformly continuous on J. Therefore, there exists $\delta \in (0,1)$ such that $x, y \in J$ and $|x-y| < \delta$ implies $|f(x)-f(y)| < \epsilon/(4N+4)$.

Let $|t| < \delta$. Then $|x| \le N+1$ implies $x - t \in J$ and $|x| > N+1$ implies $|x-t| > N$. It follows that

$$\begin{aligned}
\|g_t - f\|_{L^1(\mathbf{R})} &= \int_{-\infty}^{\infty} |f(x-t) - f(x)|dx \\
&= \int_{|x|\le N+1} |f(x-t)-f(x)|dx + \int_{|x|>N+1} |f(x-t)-f(x)|dx \\
&\le \int_{|x|\le N+1} \frac{\epsilon}{4N+4} dx + \int_{|x|>N+1} |f(x-t)|dx + \int_{|x|>N+1} |f(x)|dx \\
&\le (2N+2)\frac{\epsilon}{4N+4} + 2\int_{|x|>N} |f(x)|dx \\
&\le \frac{\epsilon}{2} + 2\frac{\epsilon}{4} = \epsilon.
\end{aligned}$$

Therefore $g_t \to f$ in $L^1(\mathbf{R})$ as $t \to 0$.

Lebesgue Integral Way: Let $\epsilon > 0$ be arbitrary. Since f may not be bounded, define

$$f_k(x) = \begin{cases} f(x) & \text{for } |f(x)| < k \\ 0 & \text{otherwise.} \end{cases}$$

Since $|f_k - f| \to 0$ pointwise and $|f_k - f| \le |f_k| + |f| \le 2|f|$ where $2|f|$ is integrable on $\mathbf{R}$, then the dominated convergence theorem implies that

$$\lim_{k\to\infty} \int_{\mathbf{R}} |f_k(x) - f(x)|dx \to \int_{\mathbf{R}} 0\, dx = 0 \qquad \text{as} \qquad k\to\infty.$$

Choose k large enough such that

$$\int_{\mathbf{R}} |f_k(x) - f(x)|dx < \frac{\epsilon}{6}.$$

Now

$$\begin{aligned}
\|g_t - f\|_{L^1(\mathbf{R})} &= \int_{\mathbf{R}} |f(x-t) - f(x)|dx \\
&\le \int_{\mathbf{R}} |f(x-t) - f_k(x-t)|dx + \int_{\mathbf{R}} |f_k(x-t) - f_k(x)|dx + \int_{\mathbf{R}} |f_k(x) - f(x)|dx \\
&= 2\int_{\mathbf{R}} |f_k(x) - f(x)|dx + \int_{\mathbf{R}} |f_k(x-t) - f_k(x)|dx \\
&\le \frac{\epsilon}{3} + \int_{\mathbf{R}} |f_k(x-t) - f_k(x)|dx.
\end{aligned}$$

We now turn our attention to showing

$$\int_{\mathbf{R}} |f_k(x-t) - f_k(x)|dx \to 0 \qquad \text{as} \qquad t \to 0.$$

Since f was $L^1(\mathbf{R})$ integrable, then f_k is $L^1(\mathbf{R})$ integrable. Choose N large enough such that

$$\int_{|x|>N} |f_k(x)|dx < \frac{\epsilon}{6}.$$

Since f was piecewise continuous then f_k is piecewise continuous. Therefore $|f_k(x-t)-f_k(x)| \to 0$ pointwise as $t \to 0$ for almost every x. Since $|f_k(x-t)-f_k(x)| \le |f_k(x-t)|+|f_k(x)| \le 2k$ where $h(x) = 2k$ is integrable on $[-N-1, N+1]$, then the dominated convergence theorem implies that

$$\lim_{t\to 0} \int_{|x|\le N+1} |f_k(x-t) - f_k(x)|dx \to \int_{|x|\le N+1} 0\,dx = 0 \qquad \text{as} \qquad t \to 0.$$

Therefore, there is $\delta \in (0,1)$ such that $|t| < \delta$ implies

$$\int_{|x|\le N+1} |f_k(x-t) - f_k(x)|dx \le \frac{\epsilon}{3}.$$

It follows that

$$\begin{aligned}\int_{\mathbf{R}} |f_k(x-t) - f_k(x)|dx &= \int_{|x|\le N+1} |f_k(x-t) - f_k(x)|dx + \int_{|x|>N+1} |f_k(x-t) - f_k(x)|dx \\ &\le \frac{\epsilon}{3} + \int_{|x|>N+1} |f_k(x-t)|dx + \int_{|x|>N+1} |f_k(x)|dx \\ &\le \frac{\epsilon}{3} + 2\int_{|x|>N} |f_k(x)|dx \le \frac{\epsilon}{3} + 2\frac{\epsilon}{6} = \frac{2\epsilon}{3}.\end{aligned}$$

This inequality together with the inequality at the bottom of the previous page implies that

$$\|g_t - f\|_{L^1(\mathbf{R})} \le \frac{1}{3} + \frac{2\epsilon}{3} = \epsilon \qquad \text{whenever} \qquad |t| < \delta.$$

Therefore $g_t \to f$ in $L^1(\mathbf{R})$ as $t \to 0$.