Theorem on Haar Coefficient Decay Rates. Let $f \in C^1([0,1])$. Then

$$\langle f, h_{j,k}\rangle = \mathcal{O}(2^{-3j/2}) \qquad \text{as} \qquad j \to \infty.$$

Proof. For $f \in C^1([0,1])$ define

$$A = \sup\big\{\, |f'(t)| : t \in [0,1] \,\big\}.$$

Given any $a \in [0,1]$ the fundamental theorem of Calculus implies

$$f(x) = f(a) + \int_a^x f'(t)\,dt.$$

Therefore,

$$\begin{aligned}
\langle f, h_{j,k}\rangle &= \int_0^1 \Big(f(a) + \int_a^x f'(t)\,dt\Big) h_{j,k}(x)\,dx \\
&= \int_0^1 \Big(\int_a^x f'(t)\,dt\Big) h_{j,k}(x)\,dx.
\end{aligned}$$

Taking $a = k/2^j$ yields

$$\begin{aligned}
\big|\langle f, h_{j,k}\rangle\big| &\le \int_0^1 \bigg|\int_{k/2^j}^x f'(t)\,dt\bigg| p_{j,k}(x)\,dx \\
&\le 2^{j/2} \int_{k/2^j}^{(k+1)/2^j} \int_{k/2^j}^x \big|f'(t)\big|\,dt\,dx \\
&\le A2^{j/2} \int_{k/2^j}^{(k+1)/2^j} \int_{k/2^j}^x dt\,dx \\
&= A2^{j/2} \int_{k/2^j}^{(k+1)/2^j} \Big(x - \frac{k}{2^j}\Big)\,dx \\
&= A2^{j/2} \int_0^{1/2^j} x\,dx = \frac{A}{2}\frac{2^{j/2}}{2^{2j}} = \frac{A}{2} 2^{-3j/2}.
\end{aligned}$$

It follows that

$$\langle f, h_{j,k}\rangle = \mathcal{O}(2^{-3j/2}) \qquad \text{as} \qquad j \to \infty.$$

This finishes the proof.