Haar Coefficient Decay Rates Case 2. This handout analyzes the decay of the Haar coefficients as $j \to \infty$ in the case where f has a jump discontinuity.

Theorem 1. If f is bounded on $[0,1]$ then $\langle f, h_{j,k}\rangle = \mathcal{O}(2^{-j/2})$ as $j \to \infty$.

Proof. Let $B = \sup\{\, |f(t)| : t \in [0,1]\,\}$. Then

$$|\langle f, h_{j,k}\rangle| = \Big|\int_0^1 f(t)h_{j,k}(t)\,dt\Big| \le \int_0^1 |f(t)||h_{j,k}(t)|\,dt$$
$$\le \int_{k/2^j}^{(k+1)/2^j} B2^{j/2}\,dt = B2^{j/2}\Big(\frac{k+1}{2^j} - \frac{k}{2^j}\Big) = B2^{-j/2}.$$

Theorem 2. Suppose f is piecewise differentiable with bounded derivative on $[0,1]$ and has a jump discontinuity at x_0 where x_0 is a dyadic irrational. Then the asymptotic bound in Theorem 1 is sharp.

Before proving Theorem 2 we prove the following lemma:

Lemma. If $x_0 \in [0,1]$ is a dyadic irrational then there is an increasing sequence j_n of natural numbers such that

$$\frac{1}{4} < 2^{j_n}x_0 \bmod 1 < \frac{1}{2}.$$

Note that $2^{j_n}x_0 \bmod 1$ is equal to $2^{j_n}x_0 - \lfloor 2^{j_n}x_0\rfloor$ where $\lfloor x\rfloor$ is the greatest integer less than or equal to x.

Proof. Write x_0 using the dyadic expansion

$$x_0 = \sum_{i=1}^{\infty}\frac{b_i}{2^i} \qquad \text{where } b_i \in \{0,1\} \text{ for all } i \in \mathbf{N}.$$

Define

$$J_0 = \{\, i : b_i = 0\,\} \qquad \text{and} \qquad J_1 = \{\, i : b_i = 1\,\}.$$

Claim that both J_0 and J_1 are infinite. If not then one must be finite. If J_1 is finite then there is some N such that $b_i = 0$ for all $i \ge N$. Then

$$x_0 = \sum_{i=1}^{\infty}\frac{b_i}{2^i} = \sum_{i=1}^{N-1}\frac{b_i}{2^i}$$

would be dyadic rational. If J_0 is finite then there is some N such that $b_i = 1$ for all $i \ge N$. In this case

$$x_0 = \sum_{i=1}^{\infty}\frac{b_i}{2^i} = \sum_{i=1}^{N-1}\frac{b_i}{2^i} + \sum_{i=N}^{\infty}\frac{1}{2^i} = \sum_{i=1}^{N-1}\frac{b_i}{2^i} + \frac{1}{2^{N-1}}$$

would again be dyadic rational. Therefore J_0 and J_1 must both be infinite.

Since both J_0 and J_1 are infinite, there must be an increasing sequence j_n such that

$$b_{j_n+1} = 0 \qquad \text{and} \qquad b_{j_n+2} = 1 \qquad \text{for every} \qquad n \in \mathbf{N}.$$

Now

$$2^{j_n}x_0 \bmod 1 = \sum_{i=1+j_n}^{\infty} 2^{j_n}\frac{b_i}{2^i} = \sum_{i=1}^{\infty}\frac{b_{j_n+i}}{2^i} = \frac{b_{j_n+1}}{2} + \frac{b_{j_n+2}}{4} + \sum_{i=3}^{\infty}\frac{b_{j_n+i}}{2^i}$$
$$= \frac{1}{4} + \sum_{i=3}^{\infty}\frac{b_{j_n+i}}{2^i} = \frac{1}{4} + \frac{1}{4}\sum_{i=1}^{\infty}\frac{b_{j_n+i+2}}{2^i} = \frac{1}{4} + \frac{1}{4}w.$$

Since x_0 is dyadic irrational then w is dyadic irrational. Therefore $0 < w < 1$ and consequently $1/4 < 2^{j_n}x_0 \bmod 1 < 1/2$. This finishes the proof of the Lemma.

Proof of Theorem 2. Since f is piecewise differentiable there are only finitely many points $E \subseteq [0,1]$ where f is not differentiable. Define

$$\delta = \min\left\{\, |x-y| : x, y \in E \text{ and } x \neq y \,\right\}.$$

By Lemma 1 there is an increasing sequence j_n of natural numbers such that

$$\frac{1}{4} < 2^{j_n}x_0 \bmod 1 < \frac{1}{2} \qquad \text{for every} \qquad n \in \mathbf{N}.$$

We can assume without loss of generality that $2^{-j_1} < \delta$. For each $n \in \mathbf{N}$ choose k_n so that

$$x_0 \in \Big[\frac{k_n}{2^{j_n}}, \frac{k_n+1}{2^{j_n}}\Big) \qquad \text{or equivalently} \qquad 2^{j_n}x_0 \in [k_n, k_n+1).$$

By definition $2^{j_n}x_0 \bmod 1 = 2^{j_n}x_0 - k_n$. Therefore

$$\frac{1}{4} < 2^{j_n}x_0 - k_n < \frac{1}{2} \qquad \text{or equivalently} \qquad x_0 \in \Big(\frac{k_n+\frac{1}{4}}{2^{j_n}}, \frac{k_n+\frac{1}{2}}{2^{j_n}}\Big).$$

For notational convenience, given $n \in \mathbf{N}$ fixed, define

$$h = h_{j_n,k_n}, \qquad j = j_n, \qquad a = \frac{k_n}{2^{j_n}}, \qquad b = \frac{k_n+1}{2^{j_n}} \qquad \text{and} \qquad c = \frac{a+b}{2}.$$

Since $b - a = 2^{-j} \leq 2^{-j_1} < \delta$ then f is differentiable on (a, x_0) and (x_0, b). For $t \in (a, x_0)$ the Fundamental Theorem of Calculus implies that

$$f(x_0^-) - f(t) = \int_t^{x_0} f'(s)\,ds$$

and for $t \in (x_0, b)$ that

$$f(t) - f(x_0^+) = \int_{x_0}^{t} f'(s)\,ds.$$

Consequently,

$$\begin{aligned}\langle f, h\rangle &= \int_0^1 f(t)h(t)\,dt = \int_a^b f(t)h(t)\,dt\\ &= \int_a^{x_0}\Big(f(x_0^-) - \int_t^{x_0} f'(s)\,ds\Big)h(t)\,dt + \int_{x_0}^{b}\Big(f(x_0^+) + \int_{x_0}^{t} f'(s)\,ds\Big)h(t)\,dt\\ &= I + J\end{aligned}$$

where

$$I = \int_a^{x_0} f(x_0^-)h(t)\,dt + \int_{x_0}^b f(x_0^+)h(t)\,dt$$

and

$$J = -\int_a^{x_0}\int_t^{x_0} f'(s)h(t)\,ds\,dt + \int_{x_0}^b\int_{x_0}^t f'(s)h(t)\,ds\,dt.$$

Estimate I from below. Recall that

$$h(t) = \begin{cases} 2^{j/2} & \text{for } t \in [a, c) \\ -2^{j/2} & \text{for } t \in [c, b) \end{cases}$$

and that $x_0 \in (a + 2^{-j-2}, c)$. Therefore

$$\int_a^{x_0} f(x_0^-)h(t)\,dt = f(x_0^-)2^{j/2}(x_0 - a)$$

and also

$$\begin{aligned}\int_{x_0}^b f(x_0^+)h(t)\,dt &= \int_{x_0}^c f(x_0^+)2^{j/2}\,dt - \int_c^b f(x_0^+)2^{j/2}\,dt \\ &= f(x_0^+)2^{j/2}(c - x_0 - b + c) = -f(x_0^+)2^{j/2}(x_0 - a).\end{aligned}$$

It follows that

$$I = \big(f(x_0^-) - f(x_0^+)\big)2^{j/2}(x_0 - a)$$

and hence

$$|I| \geq \big|f(x_0^-) - f(x_0^+)\big|2^{j/2}2^{-j-2} = \frac{1}{4}\big|f(x_0^-) - f(x_0^+)\big|2^{-j/2}.$$

Now estimate J from above. Since f' is bounded $M = \sup\{\,|f'(t)| : t \in [0,1] \setminus E\,\}$ is finite. It follows that

$$\begin{aligned}|J| &\leq \int_a^{x_0}\int_t^{x_0} M2^{j/2}\,ds\,dt + \int_{x_0}^b\int_{x_0}^t M2^{j/2}\,ds\,dt \\ &\leq 2M2^{j/2}\int_a^b\int_a^b ds\,dt = 2M2^{j/2}(b-a)^2 = 2M2^{-3j/2}.\end{aligned}$$

Therefore, for every $n \in \mathbf{N}$ we have

$$|\langle f, h_{j_n,k_n}\rangle| = |I + J| \geq |I| - |J| \geq \frac{1}{4}\big|f(x_0^-) - f(x_0^+)\big|2^{-j_n/2} - 2M2^{-3j_n/2}.$$

Choose N so large that $n \geq N$ implies

$$2M2^{-3j_n/2} < \frac{1}{8}\big|f(x_0^-) - f(x_0^+)\big|2^{-j_n/2}.$$

Then, for $n \geq N$ it follows that

$$|\langle f, h_{j_n,k_n}\rangle| \geq A2^{-j_n/2} \qquad \text{where} \qquad A = \frac{1}{8}\big|f(x_0^-) - f(x_0^+)\big|.$$

This shows the bound in Theorem 1 is sharp and finishes the proof of Theorem 2.