Lemma 1 If $x_0 \in [0,1)$ is dyadic irrational then there is a subsequence j_n of natural numbers such that

$$\frac{1}{4} \le 2^{j_n} x_0 \mod 1 \le \frac{1}{2}.$$

proof: Write x_0 as its dyadic expansion

$$x_0 = \sum_{i=1}^{\infty} \frac{b_i}{2^i} \quad \text{where} \quad b_i \in \{0,1\} \text{ for all } i.$$

Define

$$J_0 = \{i : b_i = 0\} \quad \text{and} \quad J_1 = \{i ; b_i = 1\}.$$

Claim both sets are infinite. If not then x_0 would be a dyadic rational, either because the dyadic expansion terminates or because is would end with repeating 1's.

Therefore, there is an infinite sequence j_n such that

$$b_{j_n+1} = 0 \quad \text{and} \quad b_{j_n+2} = 1.$$

Now

$$2^{j_n} x_0 \mod 1 = \sum_{i=1+j_n}^{\infty} 2^{j_n} \frac{b_i}{2^i} = \sum_{l=1}^{\infty} \cancel{2^{j_n}} \frac{b_{l+j_n}}{2^{l+\cancel{j_n}}}$$

$$(l = i - j_n) \qquad i = l + j_n$$

$$= \frac{b_{1+j_n}}{2} + \frac{b_{2+j_n}}{4} + \sum_{l=3}^{\infty} \frac{b_{l+j_n}}{2^l} =$$

$$= \frac{0}{2} + \frac{1}{4} + \frac{1}{4} \boxed{\sum_{m=1}^{\infty} \frac{b_{m+2+j_n}}{2^m}} \in [0,1)$$

~~Then~~ Since $0 \le \sum_{m=1}^{\infty} \frac{b_{m+2+j_n}}{2^m} \le 1$ then

$$\frac{1}{4} \le 2^{j_n} x_0 \mod 1 \le \frac{1}{2}.$$

Theorem: If f is bounded on $[0,1)$, then

$$|\langle f, h_{j,k}\rangle| = O(2^{-j/2}).$$

Proof. Let $B = \sup\{|f(t)| : t \in (0,1)\}$. Then

$$|\langle f, h_{j,k}\rangle| = \left|\int_0^1 f(t) h_{j,k}(t)\,dt\right| \le \int_0^1 |f(t)||h_{j,k}(t)|\,dt$$

$$\le B \int_{\frac{k}{2^j}}^{\frac{k+1}{2^j}} 2^{j/2}\,dt = B\,2^{j}\left(\frac{k+1}{2^j} - \frac{k}{2^j}\right) = B\,2^{-j/2}.$$

Theorem: Suppose f is piecewise differentiable ~~with bounded derivative~~ and has a jump discontinuity at x_0 where x_0 is a dyadic irrational. Then the asymptotic bound in the above theorem is sharp.

Proof: By the lemma there is a subsequence j_n such that $\frac{1}{4} \le 2^{j_n} x_0 \bmod 1 \le \frac{1}{2}$ for $n = 1, 2, 3, \ldots$. Choose k_n so that $x_0 \in \left[\frac{k_n}{2^{j_n}}, \frac{k_n+1}{2^{j_n}}\right)$. Then we that $2^{j_n} x_0 \in [k_n, k_n+1)$, so $2^{j_n} x_0 - k_n = 2^{j_n} x_0 \bmod 1$.
Therefore $\frac{1}{4} \le 2^{j_n} x_0 - k_n \le \frac{1}{2}$, or $\frac{k_n + \frac{1}{4}}{2^{j_n}} \le x_0 \le \frac{k_n + \frac{1}{2}}{2^{j_n}}$.
Since x_0 is dyadic irrational then $x_0 \in \left(\frac{k_n+\frac{1}{4}}{2^{j_n}}, \frac{k_n + \frac{1}{2}}{2^{j_n}}\right)$ and not equal to the end points. $= (a + \frac{1}{2^{j_n+2}}, c)$

Choose N so large that $n \ge N$ implies that on the interval $\left[\frac{k_n}{2^{j_n}}, \frac{k_n+1}{2^{j_n}}\right]$ the only place where f fails to have a derivative is at x_0. We can do this because f is piecewise differentiable so there are only finitely many places where f fails to have a derivative.

Given $n \geq N$ let $[a,b] = \left[\frac{k_n}{2^{j_n}}, \frac{k_n+1}{2^{j_n}}\right]$. Then for $t \in [a, x_0]$ then

$$f(x_0^-) - f(t) = \int_t^{x_0} f'(s)\,ds.$$

for $t \in [x_0, b]$ then

$$f(t) - f(x_0^+) = \int_{x_0}^t f'(s)\,ds.$$

Consequently,

$$\langle f, h_{j_n,k_n} \rangle = \int_0^1 f(t)\, h_{j_n,k_n}(t)\,dt = \int_a^b f(t)\, h_{j_n,k_n}(t)\,dt$$

$$= \int_a^{x_0} \left(f(x_0^-) - \int_t^{x_0} f'(s)\,ds \right) h_{j_n k_n}(t)\,dt$$

$$+ \int_{x_0}^b \left(f(x_0^+) + \int_{x_0}^t f'(s)\,ds \right) h_{j_n k_n}(t)\,dt$$

$$= I_n + J_n$$

where

$$I_n = \int_a^{x_0} f(x_0^-)\, h_{j_n k_n}(t)\,dt + \int_{x_0}^b f(x_0^+)\, h_{j_n k_n}(t)\,dt$$

and

$$J_n = -\int_a^{x_0} \int_t^{x_0} f'(s)\,ds\, h_{j_n k_n}(t)\,dt + \int_{x_0}^b \int_{x_0}^t f'(s)\,ds\, h_{j_n k_n}(t)\,dt$$

Estimate I_n from below. By choice of the j_n's we have that $x_0 \in [a, c)$ where $c = \frac{a+b}{2}$. Therefore $h_{j_n k_n}(t) = 2^{j_n/2}$ for all $t \in [a, x_0)$. So

$$\int_a^{x_0} f(x_0^-)\, h_{j_n k_n}(t)\,dt = 2^{j_n/2} (x_0 - a)\, f(x_0^-).$$

also

$$\int_{x_0}^{b} f(x_0^+)\, h_{j_n k_n}(t)\, dt = \int_{x_0}^{c} f(x_0^+)\, 2^{j_n/2}\, dt - \int_{c}^{b} f(x_0^+)\, 2^{j_n/2}\, dt$$

$$= f(x_0^+)\, 2^{j_n/2}(c - x_0) - f(x_0^+)\, 2^{j_n/2}(b-c)$$

$$= f(x_0^+)\, 2^{j_n/2}(2c - b - x_0)$$

$$= f(x_0^+)\, 2^{j_n/2}(a + \cancel{b} - \cancel{b} - x_0)$$

$$= f(x_0^+)\, 2^{j_n/2}(a - x_0) = - f(x_0^+)\, 2^{j_n/2}(x_0 - a),$$

Therefore

$$I_n = 2^{j_n/2}\left(f(x_0^-) - f(x_0^+)\right)(x_0 - a)$$

Since $x_0 \in \left(a + \frac{1}{2^{j_n+2}}, c\right)$ then $x_0 - a > \frac{1}{2^{j_n+2}}$. Thus

$$|I_n| \geq 2^{j_n/2}\,|f(x_0^-) - f(x_0^+)|\, \frac{1}{2^{j_n+2}}$$

$$= \frac{1}{4}\,|f(x_0^-) - f(x_0^+)|\, 2^{-j_n/2}.$$

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<u>Estimate</u> J_n from above. Let $M = \sup\left\{ |f'(x)| : \left[\frac{k_N}{2^{J_N}}, \frac{k_N+1}{2^{J_N}}\right]\right\}$

$$|J_n| \leq \int_a^{x_0}\int_t^{x_0} M\, ds\, 2^{J_n/2}\, dt + \int_{x_0}^{b}\int_{x_0}^{t} M\, 2^{J_n/2}\, ds\, dt$$

$$\leq 2\int_a^b \int_a^b M\, 2^{J_n/2}\, ds\, ds = 2(b-a)^2 M\, 2^{j_n/2}$$

$$= 2\left(\frac{k_n+1}{2^{j_n}} - \frac{k_n}{2^{j_n}}\right)^2 M\, 2^{j_n/2} = 2M\, 2^{-3j_n/2}$$

$$= O(2^{-3J_n/2}).$$

Therefore

$$|\langle f, h_{j_n,k_n}\rangle| \geq |I_n| - |J_n|$$

$$\geq \frac{1}{4}|f(x_0^-)-f(x_0^+)|2^{-j_n/2} - 2M2^{-3j_n/2}$$

for j_n large enough there is a positive constant A such that

$$|\langle f, h_{j_n,k_n}\rangle| \geq A\,2^{-j_n/2}.$$

This shows the bound is assymtotically sharp.