Lemma 5.16 (Splitting Lemma)

$\text{span}\{p_{j+1,k} : k=0,1,\dots,2^{j+1}-1\} = \text{span}\{p_{j,k} : k=0,1,\dots,2^j-1\} \cup \{h_{j,k} : k=0,1,\dots,2^j-1\}.$

Proof: Let $A_j = \{p_{j,k} : j=0,1,\dots,2^j-1\}$ and $B_j = \{h_{j,k} : j=0,1,\dots,2^j-1\}$.

"$\supseteq$" Since $p_{j,k} = \frac{1}{\sqrt{2}}(p_{j+1,2k} + p_{j+1,2k+1})$ and $h_{j,k} = \frac{1}{\sqrt{2}}(p_{j+1,2k} - p_{j+1,2k+1})$

Then $\text{span}(A_j \cup B_j) \subseteq \text{span}(A_{j+1})$.

"$\subseteq$" Suppose $g \in \text{span}\, A_{j+1}$. Then there are α_j such that

$$g = \sum_{l=0}^{2^{j+1}-1} \alpha_l p_{j+1,l} = \sum_{l \text{ even}} \alpha_l p_{j+1,l} + \sum_{l \text{ odd}} \alpha_l p_{j+1,l}$$

$$= \sum_{k=0}^{2^j-1} \alpha_{2k} p_{j+1,2k} + \sum_{k=0}^{2^j-1} \alpha_{2k+1} p_{j+1,2k+1}$$

$$= \sum_{k=0}^{2^j-1} (\alpha_{2k} p_{j+1,2k} + \alpha_{2k+1} p_{j+1,2k+1})$$

$$= \sum_{k=0}^{2^j-1} \left(\frac{\alpha_{2k}+\alpha_{2k+1}}{2}(p_{j+1,2k} + p_{j+1,2k+1}) + \frac{\alpha_{2k}-\alpha_{2k+1}}{2}(p_{j+1,2k} - p_{j+1,2k+1})\right)$$

$$= \sum_{k=0}^{2^j-1} \frac{\alpha_{2k}+\alpha_{2k+1}}{\sqrt{2}} p_{j,k} + \sum_{k=0}^{2^j-1} \frac{\alpha_{2k}-\alpha_{2k}}{\sqrt{2}} h_{j,k} \in \text{span}\, A_j \cup B_j$$

Therefore $\text{span}\, A_{j+1} \subseteq \text{span}\, A_j \cup B_j$

It follows that $\text{span}\, A_{j+1} = \text{span}\, A_j \cup B_j$