Lemma 5.23 (Density of the Haar system in $C([0,1])$.)

Given $f \in C([0,1])$ and $\varepsilon > 0$ there is J and a function

$$g \in \operatorname{span}\{ p_{J,k} : k = 0, 1, \dots, 2^J - 1 \}$$

such that $\|f - g\|_{L^\infty([0,1])} < \varepsilon$.

Proof: Let $f \in C([0,1])$ and $\varepsilon > 0$. Since $[0,1]$ is a closed bounded interval then f is uniformly continuous on $[0,1]$. Therefore there is $\delta > 0$ such that $x, y \in [0,1]$ with $|x-y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon$. Choose J so large that $\frac{1}{2^J} < \delta$.

Define $\alpha_k = 2^{-J/2} f(k/2^J)$ for $k = 0, 1, \dots, 2^J - 1$.

Then $g = \sum_{k=0}^{2^J-1} \alpha_k p_{J,k} \in \operatorname{span}\{ p_{J,k} : k = 0, 1, \dots, 2^J - 1 \}$.

Claim $\|f - g\|_{L^\infty([0,1])} < \infty$.

Note that $g \in \operatorname{span}\{ p_{J,k} : k = 0, 1, \dots, 2^J - 1 \}$ implies $g(1) = 0$ whereas $f(1)$ could be anything. Generaly the L^∞ norm is defined for non-continuous functions h as

$$\|h\|_{L^\infty([0,1])} = \operatorname{ess\,sup} \{ |h(x)| : x \in [0,1] \}$$

where ess sup of a set A is defined as

$$\operatorname{ess\,sup} A = \inf \{ \sup(A \setminus E) : \lambda(E) = 0 \}.$$

In particular, then, by definition

$$\|f - g\|_{L^\infty([0,1])} = \|f - g\|_{L^\infty([0,1))}.$$

Let $x \in [0,1)$. There exists k_0 between 0 and 2^{J-1} such that $x \in I_{J,k_0} = [k_0/2^J, k_0+1/2^J)$. In particular $p_{J,k_0}(x) = 2^{J/2}$ and $p_{J,k}(x) = 0$ for $k \neq k_0$. Therefore

$$|f(x) - g(x)| = \Big| f(x) - \sum_{k=0}^{2^J-1} \alpha_k p_{J,k}(x) \Big| = |f(x) - \alpha_{k_0} 2^{J/2}|$$
$$= |f(x) - 2^{-J/2} f(k_0/2^J) 2^{J/2}| = |f(x) - f(k_0/2^J)| < \varepsilon$$

since $|x - k_0/2^J| \le 1/2^J < \delta$.