Theorem 5.24. For every $J \geq 0$ the scale J Haar system

$$A = \{ p_{J,k} : k = 0, 1, \dots, 2^J - 1 \} \cup \{ h_{j,k} : j \geq J,\ k = 0, 1, \dots, 2^j - 1 \}$$

is a complete orthonormal system.

Proof: By previous results we know A is orthonormal.

Define

$$A_j = \{ p_{j,k} : k = 0, 1, \dots, 2^j - 1 \}, \quad B_j = \{ h_{j,k} : k = 0, 1, \dots, 2^j - 1 \}.$$

Claim for $j \geq J$ that $\operatorname{Span} A_j \subseteq \operatorname{Span} A$.

We prove the claim by induction on j.

Base case: If $j = J$ then $A_j \subseteq A$. Thus $\operatorname{span} A_j \subseteq \operatorname{Span} A$.

Induction step: Suppose $\operatorname{Span} A_j \subseteq \operatorname{Span} A$. Then since $B_j \subseteq A$ we also have $\operatorname{Span} B_j \subseteq \operatorname{Span} A$. By the splitting theorem

$$\operatorname{Span} A_{j+1} = \operatorname{span} A_j \cup B_j = \operatorname{Span}(\operatorname{Span} A_j \cup \operatorname{Span} B_j)$$

$$\subseteq \operatorname{Span}(\operatorname{Span} A \cup \operatorname{Span} A) = \operatorname{span} A,$$

thus completing the induction.

Now, let $f \in C([0,1])$ and $\varepsilon > 0$.

Claim there exists $g \in A$ such that $\|f - g\|_{L^2([0,1])} < \varepsilon$.

By the lemma to this theorem there exists $J' \geq J$ and some $g \in \operatorname{span} A_{J'}$ such that $\|f - g\|_{L^\infty([0,1])} < \varepsilon$.

Since $\operatorname{Span} A_{J'} \subseteq \operatorname{Span} A$ then $g \in \operatorname{Span} A$. Moreover

$$\|f - g\|_{L^2([0,1])} = \sqrt{\int_0^1 |f(x) - g(x)|^2 \, dx} < \sqrt{\int_0^1 \|f - g\|^2_{L^\infty([0,1])} \, dx}$$

$$= \sqrt{\int_0^1 \varepsilon^2 \, dx} = \varepsilon$$

shows that A is complete.