Math 762 Final Version A

Formulas and Definitions

Fourier Series

$$f(x) = \sum_{n\in\mathbf{Z}} c_n e^{2\pi i n x/a}, \qquad c_n = \frac{1}{a}\int_0^a f(x) e^{-2\pi i n x/a} dx.$$

Fourier Transform

$$\hat{f}(\gamma) = \int_{\mathbf{R}} f(t) e^{-2\pi i \gamma t} dt, \qquad f(x) = \int_{\mathbf{R}} \hat{f}(\gamma) e^{2\pi i \gamma t} d\gamma.$$

Scaling Function Let ϕ be a scaling function for a multiresolution analaysis. Then $\phi \in L^2$, and $\{\, T_n\phi : n \in \mathbf{Z}\,\}$ is orthonormal and $V_0 = \overline{\text{span}}\{, T_n\phi : n \in \mathbf{Z}\,\}$.

Dilation and Translation

$$T_k f(x) = f(x-k), \qquad D_q f(x) = \sqrt{q} f(qx), \qquad T_k D_q = D_q T_{qk},$$

$$\widehat{T_k f}(\gamma) = e^{-2\pi\gamma k}\hat{f}(\gamma) \qquad \text{and} \qquad \widehat{D_q f}(\gamma) = \frac{1}{\sqrt{q}}\hat{f}(\gamma/q).$$

$$\phi_{j,k} = D_{2^j} T_k \phi(x) = \phi(2^j x - k), \qquad \psi_{j,k} = D_{2^j} T_k \psi(x) = \psi(2^j x - k).$$

Wavelet in Physical Space

$$\phi = \sum_{k\in\mathbf{Z}} h_k \phi_{j,k} \qquad \text{where} \qquad h_k = \langle \phi, \phi_{1,k}\rangle.$$

$$\psi = \sum_{k\in\mathbf{Z}} g_k \phi_{j,k} \qquad \text{where} \qquad g_k = (-1)^k \overline{h_{1-k}}.$$

Wavelet in Transform Space

$$\phi(\gamma) = m_0(\gamma/2)\phi(\gamma/2) \qquad \text{where} \qquad m_0(\gamma) = \frac{1}{\sqrt{2}}\sum_{k\in\mathbf{Z}} h_k e^{-2\pi i \gamma k}$$

$$\psi(\gamma) = m_1(\gamma/2)\phi(\gamma/2) \qquad \text{where} \qquad m_1(\gamma) = e^{-2\pi i(\gamma+1/2)}\overline{m_0(\gamma+1/2)}.$$

Orthogonal Projections onto V_j and W_j

$$V_j = \overline{\text{span}}\{\, T_n\phi_{j,n} : n \in \mathbf{Z}\,\}, \qquad W_j = \overline{\text{span}}\{\, T_n\psi_{j,n} : n \in \mathbf{Z}\,\}.$$

$$P_j f = \sum_{n\in\mathbf{Z}} \langle f, \phi_{j,n}\rangle \phi_{j,n}, \qquad \tilde{Q}_j f = \sum_{n\in\mathbf{Z}} \langle f, \psi_{j,n}\rangle \psi_{j,n}$$

Quadrature Mirror Filter Condition

$$h_k \in \ell^2, \qquad m_0(0) = 1, \qquad \text{and} \qquad |m_0(\gamma)|^2 + |m_0(\gamma+1/2)|^2 = 1.$$

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INSTRUCTIONS: Complete a total of 5 problems out of the 8 below. Clearly indicate which problems you wish graded. Work each problem on a separate sheet of paper (or more if necessary).

1. Suppose $\{T_n g : n \in \mathbf{Z}\}$ is an orthonormal system of translates, prove that

$$\sum_{n\in\mathbf{Z}} \left|\hat{g}(\gamma+n)\right|^2 = 1 \qquad \text{for all} \qquad \gamma \in \mathbf{R}.$$

2. Suppose $g \in L^2(\mathbf{R})$ and

$$\sum_{n\in\mathbf{Z}} \left|\hat{g}(\gamma+n)\right|^2 = 1 \qquad \text{for all} \qquad \gamma \in \mathbf{R},$$

prove that $\{T_n g : n \in \mathbf{Z}\}$ is an orthonormal system of translates.

3. Suppose $f \in C^0 \cap L^1$ and $\hat{f} \in L^1$, then

$$\lim_{\tau\to 0^+} \frac{1}{\tau}\int_{\mathbf{R}} f(t)e^{-\pi(x-t)^2/\tau^2}\,dt = f(x).$$

4. Suppose $f \in C^0 \cap L^1$ and $\hat{f} \in L^1$, then

$$\frac{1}{\tau}\int_{\mathbf{R}} f(t)e^{-\pi(x-t)^2/\tau^2}\,dt = \int_{\mathbf{R}} \hat{f}(\gamma)e^{-\pi\tau^2\gamma^2}e^{2\pi i\gamma x}\,d\gamma.$$

5. Suppose $f \in C^0 \cap L^1$ and $\hat{f} \in L^1$, then

$$\lim_{\tau\to 0^+}\int_{\mathbf{R}} \hat{f}(\gamma)e^{-\pi\tau^2\gamma^2}e^{2\pi i\gamma x}\,d\gamma = \int_{\mathbf{R}} \hat{f}(\gamma)e^{2\pi i\gamma x}\,d\gamma.$$

6. Show that $\{T_n\psi : n \in \mathbf{Z}\}$ is an orthonormal system of translates.

7. Show that $V_0 \perp W_0$.

8. Show that $V_1 = V_0 \oplus W_0$.