My research currently focuses on the following areas:

1. Embedding Theorems and Fractal Dimension
2. Alpha Models of Turbulence
3. Normal Forms of the Navier–Stokes Equations
4. Continuous and discrete-in-time Data Assimilation

This year my work with Abrahim Azouani and Edriss Titi on continuous data assimilation using general interpolant observables appeared in Nonlinear Science. I'm currently finishing two papers: one on the Assouad dimension and another on data assimilation of stochastically noisy data. Three more papers are in preparation.

Embedding Theorems and Fractal Dimension

My thesis was on embedding theorems and orthogonal projections. The motivation behind studying the continuity properties of orthogonal projections lies in the theory of exponential attractors developed in [2]. In particular, given an infinite dimensional dissipative evolution equation the goal is to create a system of ordinary differential equations that have the same exponential attractor. This amounts to projecting the attractor of the infinite dimensional dynamical system into a finite dimensional space and transferring the dynamics to that space by means of the inverse projection. The first result of my thesis was to extend the Hölder–Mane projection theorem in [1] from finite dimensional spaces to infinite dimensional spaces. Namely, I proved

> **Theorem**. *Let H be a real Hilbert space and $X \subset H$ be such that $\dim_{\mathrm{f}}(X) < m/2$. Then for any orthogonal projection P of rank m and $\delta > 0$ there is an orthogonal projection $\tilde{P}$ such that $\|\tilde{P} - P\| < \delta$ and $\tilde{P}|_X$ has Hölder inverse.*

This result was published with Foias in [4] and has been cited 39 times by independent researchers on MathSciNet.

The Hölder continuity of the inverse projection given in [4] and [7] is not sufficient to prove that solutions to the resulting finite system of ordinary differential equations are unique. For uniqueness it is necessary to have an inverse that is Lipschitz with at most an order 1 logarithmic correction term. Building upon my result in [8] for finite dimensional spaces, Robinson and I [9] show in infinite dimensional spaces that the set of orthogonal projections whose inverses are Lipschitz up to a logarithmic correction are prevalent in the sense of Hunt, Sauer and Yorke [6]. We do this for attractors X for which the set of differences $X - X$ has finite Assouad dimension.

Note the Assouad dimension $\dim_{\mathrm{a}}(X - X) < \infty$ for any dynamical system possessing an inertial manifold. Denote the box counting dimension by $\dim_{\mathrm{f}}$. It is well known that $\dim_{\mathrm{f}}(X - X) \le 2\dim_{\mathrm{f}}(X)$. However, a similar inequality does not, in general, hold true for the Assouad dimension. In particular, assume $X \subset H$ where H is a Hilbert space and suppose X is connected and consists of at least two points. We show in [9] that there exists a C^∞ bi-Lipschitz map $\psi\colon H \to H$ arbitrarily close to the identity such that $\dim_{\mathrm{a}}(\psi(X) - \psi(X)) = \infty$. Xander Henderson wrote his master's thesis [5] under my supervision. In that work, among other things, he shows there are self similar sets $X \subset \mathbf{R}$ such that $\dim_{\mathrm{a}}(X) = \epsilon$ but $\dim_{\mathrm{a}}(X - X) = 1$. Thus, bounds on $\dim_{\mathrm{a}}(X - X)$ are delicate and difficult to obtain even for very regular fractals.

In collaboration with Fraser, Henderson and Robinson I have recently shown in [3] that the notion of a grid-like iterated function system appearing in Henderson's thesis is related to the weak separation property of Martin Zerner [10]. This allows us to prove that if an iterated function system possesses the weak separation property then $\dim_{\rm a} X = \dim_{\rm f} X$. Moreover, if the iterated function system does not posses the weak separation property, then $\dim_{\rm a} X \geq 1$. A paper is currently in progress discussing this result.

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[10] M. Zerner, Weak separation properties for self-similar sets, *Proc. Amer. Math. Soc.* **124** (1996), no. 11, 3529=3539.

Alpha Models of Turbulence

As a student I spent half a year at the Center for Nonlinear Studies at Los Alamos National Laboratory. Here I met and worked with Shiyi Chen, Darryl Holm, Edriss Titi and Shannon Wynne along with my thesis advisor Ciprian Foias on the viscous Camassa–Holm equations, now known as the LANS-α model. Our work involved using the time independent LANS-α model as a closure for the Reynolds equations. This closure matched with experimental data and resulted in the three papers [1,2,3] with Chen, Foias, Holm, Titi and Wynne. These papers have been cited more than 200 times.

I continued to study the LANS-α model as an NSF postdoc with Edriss Titi University of California Irvine. I extended my work on Gevrey regularity of the LANS-α model to cover a family of Navier–Stokes-α like models involving fractional derivatives and published this as [4]. We also developed a simplified version of the LANS-α model similar to the regularization used by Leray in 1934 for his studies of the Navier–Stokes equations. These results along with computations and some additional scaling arguments regarding the energy spectrum were published with Cheskidov and Titi in [5].

Averaging the Navier–Stokes equations in the Lagrangian picture subject to appropriate geometric constraints and assumptions have yielded turbulence models with very interesting conservation properties. Of related interest are the Navier-Stokes-Voight equations and the LANS-α–β models. I'm currently interested in using data assimilation to study these models of turbulence computationally.

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Reaction Equations in Porous Media

My work in the area of bioremediation began with a grant from the Nevada NSF EPSCoR Advanced Computing in Environmental Sciences (ACES) program with Dr Satoko Kurita to study the bioremediation of contaminated soil. This grant supported Dr. Kurita as a research postdoc from January 2004 though December 2004. During this time we adapted the fractional order time and diffusion equations developed by Schumer, Benson, Meerschaert and Baeumer [4] with Dr. David Benson at the Desert Research Institute and Dr. Mark Meerschaert in the Mathematics Department at UNR to add dynamics for bacteria growth and and interaction with nutrients in a way consistent with the model by Dr. Kurita [3] and also [2]. In addition we developed a numerical code based on the fractional Adams method of Diethelm, Ford and Freed [1] for integrating fractional-in-time PDEs. This work is currently in progress.

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[4] R. Schumer, D. Benson, M.M. Meerschaert, B. Baeumer, Multiscaling fractional advection-dispersion equations and their solutions, *Water Resour. Res.* **39**:1022 (2003).

Normal Forms of the Navier–Stokes Equations

I have published two papers [3] and [4] on the normal forms of the Navier–Stokes equations. These works were completed with Foias, Ziane and Luan and based based on a construction by Foias and Saut in [1,2]. Using a Phragmen–Lindelof type theorem obtained by bootstrapping a theorem of F. and R. Nevanlinna we obtained recursive estimates on the norms of the components $W_n(u_0)$ of the normal form $W(u_0)$. These estimates then allowed the construction of a norm on the range of the normalization map for all $u_0 \in \mathcal{R}$, where $\mathcal{R} \subseteq H^1(\Omega)^3$ is the set of initial data that lead to regular solutions. Future work is planned to obtain estimates on the monomial terms of a specified degree in the asymptotic expansions and to further understand analyticity properties of the normalization map near the origin. We are also looking at the case with forcing.

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Continuous and Discrete-in-time Data Assimilation

My research on continuous and discrete-in-time data assimilation can be viewed in two ways: first as computational and theoretical results that provide insight on the practical problem of weather forecasting and second as a dynamical systems experiment to study non-linear dynamical systems. Charney, Halem, and Jastrow introduced in [1] the method of continuous data assimilation in which this time series is used to find a more accurate description of the current state of the atmosphere. Suppose u is given by

$$\frac{du}{dt} = \mathcal{F}(u), \qquad u(t_0) = u_0 \tag{1}$$

and the observations of u are given by the time series $p(t) = Pu(t)$ for $t \in [t_0, t_*]$ where P is an orthogonal projection. Since we don't know u_0, the idea is to solve

$$\frac{dq}{dt} = (I - P)\mathcal{F}(q + p), \qquad q(t_0) = q_0 \tag{2}$$

where q_0 is an arbitrarily chosen initial condition. Then $q+p$ represents the approximation of u given by the method of continuous data assimilation. For the model problem of the two-dimensional incompressible Navier–Stokes equations Titi and I use the theory of determining modes to find conditions on P and f such that

$$\big\| p(t) + q(t) - u(t) \big\|_{L^2} \to 0 \quad \text{as} \quad t \to \infty.$$

Our first paper [6] studies how the length scales present in the forcing affected the number of numerically determining modes, the second paper [8] studies how the Grashof number affects the number of numerically determining modes and the third [4] extends these results to the case where the observational data is available only at discrete time intervals.

A second approach to data assimilation introduced by Azouani, Olson and Titi in [9] uses feedback control. Given a system

$$\frac{du}{dt} = \mathcal{F}(u)$$

and an interpolant observable satisfying an inequality of the form

$$\|u - I_h(u)\|_{L^2}^2 \le c_1 h^2 \|u\|_{H^1}^2 + c_2 h^4 \|u\|_{H^2}^2 \tag{3}$$

an approximation v is then computed from the observations $I_h(u)$ using the equation

$$\frac{dv}{dt} = \mathcal{F}(v) + \mu\big(I_h(u) - I_h(v)\big). \tag{4}$$

This method is more general than the method described in equation (2) because the interpolation of the observations does not need to be given by an orthogonal projection nor does it need to take values in the domain of $\mathcal{F}$. For the model problem of the two-dimensional incompressible Navier–Stokes equations we prove provided h is small enough and μ appropriately chosen that

$$\big\|v(t) - u(t)\big\|_{L^2} \to 0 \quad \text{as} \quad t \to \infty.$$

The method is simple, works for a general class of observables with no additional *ad hoc* filtering and our analytical bounds on the resolution of the observations are comparable up to a logarithmic correction to previous estimates for equation (2).

Masakazu Gesho performed computations of (4) in his master's thesis under my supervision. He considered the two-dimensional incompressible Navier–Stokes equation on 2π-periodic domain Ω and took I_h to be the interpolation on nodal values given by

$$I_h(u)(x,t) = \sum_{i=1}^{N} u(x_i,t)\chi_{Q_i}(x)$$

where Q_i are disjoint squares that cover Ω and x_i are the center points of the Q_i. Gesho's computations showed there is a wide range of good values for the relaxation parameter μ and that algorithm (4) works computationally just as well as equation (2). These results are currently being prepared for publication.

The algorithm given by equation (4) also makes sense in the presence of stochastic errors. Suppose instead of observing $I_h(u)$ directly, that we know only $\tilde{I}_h$ given by

$$\tilde{I}_h(u)dt = I_h(u)dt + dW$$

where dW represents a Wiener process modelling the measurement errors. Replacing $I_h(u)$ by $\tilde{I}_h(u)$ in equation (4) we arrive at the stochastic differential equation

$$d\tilde{v} = F(\tilde{v}) + \mu(I_h(u) - I_h(\tilde{v}))dt + \mu dW$$

Let σ^2 be a parameter related to the variance of the error represented by W. I prove with Bessaih and Titi in [10] that if h is small enough, then there exists μ such that

$$\tilde{\epsilon}_{\min} = \limsup_{t\to\infty} \mathbf{E}\big[\|\tilde{v}(t) - u(t)\|_{L^2}^2\big] \le C\sigma^2.$$

Thus, the difference between the approximate solution and exact solution is within a factor of the error present in the measurements.

I developed a discrete in time data assimilation algorithm with Hayden and Titi in [4]. Let ψ_i be the eigenfunctions of the Stokes operator A with eigenvalues λ_i. Let $P_\lambda u$ be the L^2-orthogonal projection onto the space $H_i = \operatorname{span}\{\,\psi_i : \lambda_i \le \lambda\,\}$. Suppose the observational data for u is given by $P_\lambda(u(t_n))$ where $t_n = n\Delta t$. Conditions were found for the 2D NSE that guaranteed an approximating solution converged to the exact solution. A paper is in preparation by Olson and Titi [7] to cover the case of discrete in time general interpolant observables which satisfy the inequality (3). In this case the observational data is given by $I_h(u)(x,t_n)$ for $n \in \mathbf{N}$ and the algorithm given in [4] may be generalized as follows: Define $J = P_\lambda I_h$ and $E = I - J$. Then the approximation v is given by

$$\begin{cases} v_0 = Ju_0 \\ v_{n+1} = ES(\Delta t, v_n) + Ju(t_{n+1}) \\ v(t) = S(t, v_n) \quad \text{for} \quad t \in [t_n, t_{n+1}) \end{cases}$$

where S represents the solution semigroup of the 2D NSE equations. Note that taking $I_h = P_\lambda$ yields the original algorithm discussed in [4]. In [7] it is shown that for any $\Delta t > 0$ there exists h and λ such that $\|v(t) - u(t)\|_{L^2} \to 0$ as $t \to \infty$.

Three papers are in progress and an NSF grant proposal joint with Hakima Bessaih, Aseel Farhat and Mike Jolly entitled Determining Forms and Data Assimilation with Stochastically Noisy Data has been submitted and is currently pending review.

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